A Gelfand duality for continuous lattices

Ruiyuan Chen

Abstract

We prove that the category of continuous lattices and meet- and directed join-preserving maps is dually equivalent, via the hom functor to $[0, 1]$, to the category of complete Archimedean meet-semilattices equipped with a finite meet-preserving action of the monoid of continuous monotone maps of $[0, 1]$ fixing 1. We also prove an analogous duality for completely distributive lattices. Moreover, we prove that these are essentially the only well-behaved "sound classes of joins Φ, dual to a class of meets" for which "Φ-continuous lattice" and "Φ-algebraic lattice" are different notions, thus for which a 2-valued duality does not suffice.

1 Introduction

The classical Gelfand duality asserts that a compact Hausdorff space X may be recovered from its ring of continuous functions $C(X)$, and moreover such rings are up to isomorphism precisely the commutative C^*-algebras. From a categorical perspective, $C(X)$ is best regarded as having "underlying set" given by its (positive) unit ball, i.e., consisting of continuous $\mathbb{I} := [0, 1]$-valued functions, so that Gelfand duality falls under the umbrella of Stone-type dualities induced by two "commuting" structures on $\mathbb{I}$; see [Joh82, VI §4]. Namely, $\mathbb{I}$ is equipped with its usual compact Hausdorff topology, and also with all operations $\mathbb{I}^\kappa \to \mathbb{I}$ "commuting" with the topology, i.e., which are continuous. Thus, for another object in either category, the hom functor into $\mathbb{I}$ yields a dual in the other category, and this gives a dual adjunction, which Gelfand duality asserts is an equivalence. An explicit axiomatization of the dual operations on the $\mathbb{I}$-valued $C(X)$ was recently given in [MR17]; see there for a detailed history of $\mathbb{I}$-valued Gelfand duality. In [HNN18], [Abb19], $\mathbb{I}$-valued Gelfand duality was further extended to compact partially ordered spaces (*a la* Nachbin).

In this note, we prove analogous Gelfand-type dualities for compact pospaces equipped with lattice operations. Recall that a **continuous lattice** is a compact topological meet-semilattice obeying a "local convexity under meets" condition, that each point has a neighborhood basis of subsemilattices. Equivalently, they can be defined purely order-theoretically as posets with arbitrary meets distributing over directed joins. An analog of Urysohn's lemma, sometimes known as the Urysohn–Lawson lemma, states that every continuous lattice X admits enough morphisms to $\mathbb{I}$, i.e., the canonical evaluation map $X \to \mathbb{I}^{\mathrm{Hom}(X,\mathbb{I})}$ is an embedding; see [G$^+$03, IV-3.3], [Joh82, VII 3.2]. It is thus natural to ask whether, by equipping $\mathrm{Hom}(X, \mathbb{I})$ with suitable structure commuting with the continuous lattice structure on $\mathbb{I}$, we may recover X as the double dual.

Let $\widehat{\mathbb{U}}$ denote the monoid of continuous monotone maps $\mathbb{I} \to \mathbb{I}$ fixing 1, i.e., all unary operations on $\mathbb{I}$ commuting with the continuous lattice structure. Note that finite meets do as well. By a

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$\widehat{\mathbb{U}}$**-module**, we mean a unital meet-semilattice equipped with an action of $\widehat{\mathbb{U}}$ preserving finite meets in both variables. In every $\widehat{\mathbb{U}}$-module A, we have a canonical pseudoquasimetric

$$\rho(a,b) := \bigwedge\{r \in \mathbb{I} \mid a \le b \dotplus r\}$$

where $b \dotplus r$ denotes the result of the action on b of the truncated addition $(-) \dotplus r \in \widehat{\mathbb{U}}$. We say A is **Archimedean** if $\rho(a,b) = 0 \implies a \le b$, and **complete** if A is Archimedean and complete with respect to the induced metric $d(a,b) := \rho(a,b) \vee \rho(b,a)$. We prove

Theorem 1.1 (Corollary 5.9)**.** *Hom into $\mathbb{I}$ yields a dual equivalence of categories between continuous lattices and complete $\widehat{\mathbb{U}}$-modules.*

There is a generalization of continuous lattice theory, with the role of directed joins replaced by an arbitrary "class of joins Φ" obeying suitable axioms; see [WWT78], [BE83], [Xu95], as well as [AK88], [ABLR02], [KS05] for a further extension in enriched category theory. Other than $\Phi =$ "directed joins", the most well-known case is $\Phi =$ "all joins", for which Φ-continuous lattices are completely distributive lattices. As for continuous lattices, there is a Urysohn-type lemma, stating that all completely distributive lattices admit enough morphisms to $\mathbb{I}$; see [G^+03, IV-3.31–32], [Joh82, 1.10–14]. We likewise boost this to a Gelfand-type duality as follows.

Let $\mathbb{U} \subseteq \widehat{\mathbb{U}}$ denote the monoid of complete lattice morphisms, i.e., monotone surjections. A $\mathbb{U}$**-poset** is a poset with a monotone action of $\mathbb{U}$. There is a canonical way of defining a pseudoquasimetric on a $\mathbb{U}$-poset, agreeing with the above definition in $\widehat{\mathbb{U}}$-modules; see Definition 4.2. A $\mathbb{U}$-poset A is **stackable** if, intuitively speaking, an element $a \in A$ may be specified via its "restrictions to sublevel and superlevel sets $a^{-1}([0,r]), a^{-1}([r,1])$" for any $0 < r < 1$; see Definition 4.12.

Theorem 1.2 (Corollary 5.5)**.** *Hom into $\mathbb{I}$ yields a dual equivalence of categories between completely distributive lattices and complete stackable $\mathbb{U}$-posets.*

In fact, we prove a single result underlying Theorems 1.1 and 1.2, for a "class of joins Φ dual to a class of meets Ψ^{op}", more precisely for a *sound* class of joins in the sense of [ABLR02], [KS05]; see Section 3. This general result, Theorem 5.2, says that Φ-continuous lattices are dual to complete stackable $\mathbb{U}$-Ψ^{op}-inflattices, *provided that not all Φ-continuous lattices are Φ-algebraic*, i.e., already admit enough morphisms into 2. This is a reasonable restriction, since for these other Φ, we instead have a simple 2-valued duality generalizing the classical Hofmann–Mislove–Stralka duality [HMS74] between algebraic lattices and meet-semilattices (see Corollary 3.7).

Part of the reason we work with general Φ is to hint at the possibility of generalizing to quantale-enriched posets, or even to enriched categories, which we plan to pursue in future work. However, in the original context of mere posets, it turns out that essentially the only Φ are the classical ones:

Theorem 1.3 (Theorem 3.9)**.** *There are precisely 4 sound classes of joins Φ for which not every Φ-continuous lattice is Φ-algebraic: "directed joins", "all joins", and the minor variations including/excluding empty joins.*

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2 Φ-continuous lattices

We assume familiarity with basic category theory. For a category C, $\mathsf{C}(X, Y)$ will denote the hom-set of morphisms from X to Y, while C^{op} will denote the opposite category; this includes opposite posets. We let Pos denote the category of posets, Sup denote the category of suplattices (i.e., complete lattices with join-preserving maps as morphisms), Inf denote the category of inflattices, and $\mathsf{CLat} = \mathsf{Sup} \cap \mathsf{Inf}$ denote the category of complete lattices. These are all locally ordered categories: each hom-set is partially ordered pointwise, and composition is monotone on both sides. For $f : X \to Y \in \mathsf{Pos}$ left adjoint to $g : Y \to X$, we will write $f = g^+$ and $g = f^\times$. We will frequently use the "mate calculus": for monotone h, k, we have $h \circ g \le k \iff h \le k \circ f$.

For a poset X, we let $\mathcal{L}(X)$ denote the poset of lower sets $\phi \subseteq X$, ordered via $\subseteq$. Then $\mathcal{L} : \mathsf{Pos} \to \mathsf{Pos}$ is the free suplattice monad, where the monad structure consists of:

- unit $\downarrow = \downarrow_X : X \to \mathcal{L}(X)$, where $\downarrow x = \{y \in X \mid y \le x\}$ is the principal ideal below x;
- multiplication $\bigcup : \mathcal{L}(\mathcal{L}(X)) \to \mathcal{L}(X)$;
- $f : X \to Y \in \mathsf{Pos}$ inducing $f_* = \mathcal{L}(f) : \mathcal{L}(X) \to \mathcal{L}(Y) \in \mathsf{Sup}$, where $f_*(\phi) = \bigcup_{x \in \phi} \downarrow f(x)$.

We now review the theory of "relative" suplattices for a "class of joins" Φ. This is a special case of the theory of "classes of colimits" in enriched category theory [AK88], [ABLR02], [KS05], and has also been well-studied in the order theory literature as "Z-completeness" [WWT78], [BE83]. We will use notation and terminology based on that from enriched categories.

Definition 2.1. A **join doctrine** is a class Φ of posets ϕ, thought of as indexing posets for certain joins $\bigvee_{x \in \phi} f(x)$ of monotone $f : \phi \to Y$. We require Φ to obey the following "saturation" conditions:

(i) The singleton poset $\mathbf{1}$ is in Φ.

(ii) If ϕ is a poset which is a union $\bigcup \Psi$ of a set $\Psi \subseteq \Phi$ of subposets $\psi \subseteq \phi$ which are in Φ, and also Ψ (as a poset under $\subseteq$) is in Φ, then $\phi \in \Phi$.

(iii) If $f : \phi \to \psi$ is a monotone map with cofinal image, and $\phi \in \Phi$, then $\psi \in \Phi$.

(iv) If $\phi \subseteq \psi$ is a cofinal subposet, and $\psi \in \Phi$, then $\phi \in \Phi$.

A **Φ-join** in a poset X is a join of a subset $\phi \subseteq X$ such that $\phi \in \Phi$. A **Φ-suplattice** is a poset with all Φ-joins; we denote the category of all such (and monotone Φ-join-preserving maps) by $\Phi\mathsf{Sup}$. A **Φ-ideal** in a Φ-suplattice is a lower sub-Φ-suplattice. The **free Φ-suplattice** generated by a poset X is the subset $\Phi(X) \subseteq \mathcal{L}(X)$ of all lower subsets of X in Φ. Note that for a poset ϕ, we have $\phi \in \Phi \iff \phi \in \Phi(\phi)$; we thereby identify the class of posets Φ with the submonad $\Phi \subseteq \mathcal{L}$.

Example 2.2.

- The "class of directed joins" is given by the join doctrine $\Phi :=$ all directed posets, for which a Φ-suplattice is a directed-complete poset (DCPO), a Φ-ideal is a Scott-closed subset, and $\Phi(X)$ is the ideal completion of X (note: *not* "Φ-ideal completion").
- The "class of finite joins" is given by $\Phi :=$ all posets with finite cofinality.
- The "class of all joins" is given by $\Phi :=$ all posets.
- The least join doctrine, of "trivial joins", is given by $\Phi :=$ posets with a greatest element.

Remark 2.3. In [AK88] and [KS05], a more general notion of "class of colimits" is considered, consisting in the posets case of an arbitrary submonad $\Phi \subseteq \mathcal{L}$, i.e., an assignment to each poset X of a set of lower sets $\Phi(X) \subseteq \mathcal{L}(X)$ closed under the monad operations on $\mathcal{L}$.

The precise connection with our definition of "join doctrine" as a class of posets is as follows. Each join doctrine Φ induces a free Φ-suplattice submonad as above; this yields an order-embedding

$$\{\text{join doctrines}\} \hookrightarrow \{\text{submonads of } \mathcal{L}\},$$

whose image consists of those submonads $\Phi \subseteq \mathcal{L}$ obeying the additional "saturation" condition

(∗) for each order-embedding between posets $f : X \hookrightarrow Y$, we have $\Phi(X) = f_*^{-1}(\Phi(Y))$.

This condition is implied by condition (iv) in Definition 2.1 of join doctrine, and conversely, ensures that $\{\phi \in \mathsf{Pos} \mid \phi \in \Phi(\phi)\}$ is a join doctrine inducing the submonad Φ.

An example of a submonad not obeying (∗) is $\Phi(X) := \{\phi \in \mathcal{L}(X) \mid \phi \text{ has an upper bound in } X\}$, which yields the "class of bounded joins". However, (∗) is automatic for the Φ suitable for our duality purposes, which is why we use the simpler definition of "join doctrine"; see Remark 3.2.

Definition 2.4. Let Φ be a join doctrine, X be a Φ-suplattice. We define, for $x, y \in X$,

$$\begin{aligned} \twoheaddownarrow = \twoheaddownarrow^\Phi_X : X &\longrightarrow \mathcal{L}(X) \\ x &\longmapsto \bigcap\{\phi \in \Phi(X) \mid x \le \bigvee \phi\}, \\ x \ll y \;:\Longleftrightarrow\; x &\ll^\Phi y \;:\Longleftrightarrow\; x \in \twoheaddownarrow y. \end{aligned}$$

We call $x \in X$ **Φ-compact** (Φ-*atomic* in [KS05]) if $x \ll^\Phi x$, i.e., whenever $\bigvee_i y_i$ is a Φ-join $\ge x$, then some $y_i \ge x$, i.e., the indicator function of $\uparrow x : X \to 2$ preserves Φ-joins. Denote these by

$$X_\Phi := \{x \in X \mid x \ll^\Phi x\}.$$

We call X **Φ-algebraic** if it is generated under Φ-joins by $X_\Phi \subseteq X$. In that case, it is easy to see that in fact, for each $x \in X$ the set $X_\Phi \cap \downarrow x$ belongs to $\Phi(X_\Phi)$ and has join x; and this yields an order-isomorphism $X \cong \Phi(X_\Phi)$. Conversely, for any poset Y, we easily have that $\Phi(Y)$ is Φ-algebraic, with $\Phi(Y)_\Phi = \{\text{principal ideals}\} \cong Y$.

Proposition 2.5. *Let Φ be a join doctrine, X be a Φ-suplattice. The following are equivalent:*

(i) For each $x \in X$, there is a $\phi \in \Phi(X)$ such that $\phi \subseteq \twoheaddownarrow x$ and $x \le \bigvee \phi$, whence in fact $\phi = \twoheaddownarrow x$.

(ii) $\bigvee : \Phi(X) \to X$ has a left adjoint, namely $\twoheaddownarrow$.

If X is a complete lattice, these are further equivalent to:

(iii) $\bigvee : \Phi(X) \to X$ preserves meets.

(iv) Arbitrary meets distribute over Φ-joins: if $\bigvee_{j \in J_i} x_{i,j}$ is a Φ-join for each $i \in I$, then

$$\bigwedge_{i \in I} \bigvee_{j \in J_i} x_{i,j} = \bigvee_{(j_i)_i \in \prod_i J_i} \bigwedge_{i \in I} x_{i,j_i}.$$

All of these hold if X is algebraic, with $\twoheaddownarrow = \downarrow_* : \Phi(X_\Phi) \to \Phi(\Phi(X_\Phi))$, i.e.,

$$x \ll y \iff \exists z \in X_\Phi \, (x \le z \le y).$$

If (i), (ii) hold for a Φ-suplattice X, we call X **Φ-continuous**. If furthermore X is a complete lattice, we call X a **Φ-continuous lattice**, or a **Φ-algebraic lattice** if X is algebraic.

Proof. (i) $\iff$ (ii) since it is easily seen that ϕ in (i) must be $\twoheaddownarrow x$.

(ii) $\iff$ (iii) by the adjoint functor theorem.

(iii) $\iff$ (iv) because the latter says $\bigwedge_{i \in I} \bigvee \bigcup_{j \in J_i} {\downarrow} x_{i,j} = \bigvee \bigcap_{i \in I} \bigcup_{j \in J_i} {\downarrow} x_{i,j}$. $\square$

Proposition 2.6. *In every Φ-suplattice,*

(a) $\twoheaddownarrow x \subseteq {\downarrow} x$, *i.e.,* $y \ll x \implies y \le x$.

(b) $x' \le x \ll y \le y' \implies x' \ll y'$.

In a Φ-continuous Φ-suplattice,

(c) (interpolation) $\twoheaddownarrow = \bigcup \twoheaddownarrow_* \twoheaddownarrow$, *i.e.,* $\twoheaddownarrow x = \bigcup_{y \ll x} \twoheaddownarrow y$, *i.e.,*

$$z \ll x \iff \exists y\, (z \ll y \ll x).$$

Proof. The first two are obvious. For interpolation: since X is an algebra of the monad Φ, we have $\bigvee \bigcup = \bigvee \bigvee_* : \Phi(\Phi(X)) \to X$; taking left adjoints yields ${\downarrow}_* \twoheaddownarrow = \twoheaddownarrow_* \twoheaddownarrow$; now take $\bigcup$. $\square$

A **morphism of Φ-continuous lattices** is a meet-preserving, Φ-join-preserving map between Φ-continuous lattices. Let $\Phi\mathsf{CtsLat}$ denote the category of Φ-continuous lattices and morphisms, and $\Phi\mathsf{AlgLat} \subseteq \Phi\mathsf{CtsLat}$ denote the full subcategory of Φ-algebraic lattices.

Proposition 2.7. *Let $f : X \to Y$ be a right adjoint between Φ-continuous Φ-suplattices, with left adjoint $f^+ : Y \to X$. Then f preserves Φ-joins iff f^+ preserves $\ll$. Thus*

$$\begin{aligned} \Phi\mathsf{CtsLat}(X, Y)^{\mathsf{op}} &\cong {\ll}^{\Phi}\mathsf{Sup}(Y, X) := \{f^+ : Y \to X \mid f^+ \text{ preserves } \ll, \textstyle\bigvee\} \\ f &\mapsto f^+. \end{aligned}$$

Proof. $f \bigvee = \bigvee f_* : \Phi(X) \to Y$ iff, taking left adjoints, $\twoheaddownarrow f^+ = (f^+)_* \twoheaddownarrow : Y \to \Phi(X)$. $\square$

Proposition 2.8. *Let Φ be a join doctrine. The following are equivalent:*

(i) For every complete lattice X, $\Phi(X) \subseteq \mathcal{L}(X)$ is closed under meets.

(ii) For every poset X, $\Phi(\mathcal{L}(X)) \subseteq \mathcal{L}(\mathcal{L}(X))$ is closed under meets.

(iii) For every poset X, $\mathcal{L}(X)$ is Φ-continuous.

If these conditions hold, we call Φ a **continuous** join doctrine.

Proof. (i) $\implies$ (ii) is obvious.

(ii) $\implies$ (iii) since $\bigcup : \Phi(\mathcal{L}(X)) \to \mathcal{L}(X)$ is the composite of the inclusion $\Phi(\mathcal{L}(X)) \hookrightarrow \mathcal{L}(\mathcal{L}(X))$ and $\bigcup : \mathcal{L}(\mathcal{L}(X)) \to \mathcal{L}(X)$, which both preserve meets, i.e., have left adjoints.

(iii) $\implies$ (i) since the composite $\mathcal{L}(X) \xrightarrow{\twoheaddownarrow_{\mathcal{L}(X)}} \Phi(\mathcal{L}(X)) \xrightarrow{\bigvee_*} \Phi(X)$ yields the $\Phi(X)$-closure of each lower set ψ: we have $1_{\mathcal{L}(X)} \le \bigvee_* \twoheaddownarrow_{\mathcal{L}(X)}$ because $\bigcup \le \bigvee_* : \Phi(\mathcal{L}(X)) \to \Phi(X) \subseteq \mathcal{L}(X)$, while $\bigvee_* \twoheaddownarrow_{\mathcal{L}(X)}$ restricted to $\Phi(X) \subseteq \mathcal{L}(X)$ becomes $\bigvee_* {\downarrow}_* = 1_{\Phi(X)}$. $\square$

The following are the two main examples of continuous join doctrines:

Example 2.9. If Φ is the "class of directed joins", i.e., the class of all directed posets, so that $\Phi(X)$ for $X \in \mathsf{Pos}$ is the ideal completion of X, then $\ll$ is the classical way-below relation, and Φ-continuity and Φ-algebraicity become classical continuity and algebraicity for DCPOs.

Similarly, for any infinite regular cardinal κ, one can consider κ-directed joins. But it turns out that for uncountable κ, continuity and algebraicity coincide; see Corollary 2.13.

Example 2.10. If Φ is the "class of all joins", i.e., the class of all posets, so that $\Phi(X) = \mathcal{L}(X)$, then a Φ-continuous lattice is a completely distributive lattice, and $\ll$ is the "way-way-below" relation sometimes denoted $\lll$; see e.g., [G+03, IV-3.31].

Minor variations are to include/exclude empty joins, which only affects Φ-compactness of $\bot$.

Example 2.11 (the unit interval)**.** For any join doctrine Φ, $\mathbb{I} := [0, 1]$ is a Φ-continuous lattice. Indeed, $\ll$ contains $<$, since any $\phi \in \mathcal{L}(\mathbb{I})$ with $r \le \bigvee \phi$ must clearly contain $[0, r)$; thus $r = \bigvee \twoheaddownarrow r$.

We now completely characterize the $\ll^\Phi$ relation on $\mathbb{I}$, by determining which $r \in \mathbb{I}$ are Φ-compact.

Proposition 2.12. *Let Φ be a join doctrine.*

(a) For every Φ-suplattice X, $\bot \in X$ is Φ-compact iff $\varnothing \notin \Phi$. In particular, this holds for $0 \in \mathbb{I}$.

(b) If $\omega \in \Phi$ (where ω has the usual linear order), then no $r > 0$ is Φ-compact in $\mathbb{I}$. Otherwise:

- *(i) For every $\phi \in \Phi$ and $x_0, x_1, \dots \in \phi$, there are $i_0 < i_1 < \cdots$ such that $x_{i_0}, x_{i_1}, \dots$ have an upper bound in ϕ. In particular, every $x_0 \le x_1 \le \cdots \in \phi$ has an upper bound.*
- *(ii) Every Φ-continuous Φ-suplattice X which also has countable increasing joins is Φ-algebraic, with the join of any $x_0 \ll x_1 \ll \cdots \in X$ being Φ-compact. In particular, every $r > 0$ is Φ-compact in $\mathbb{I}$.*

Proof. (a) is clear from the definition of Φ-compact.

(b) If $\omega \in \Phi$, then no $r > 0$ is Φ-compact, since r is the join of a sequence in $[0, r)$. Now suppose $\omega \notin \Phi$. Then for $\phi \in \Phi$ and $x_0, x_1, \dots \in \phi$, if no infinite subfamily has an upper bound, then we have a monotone map $\phi \to \omega$ taking $\phi \setminus \bigcup_n \uparrow x_n$ to 0 and each $\uparrow x_n \setminus \bigcup_{m>n} \uparrow x_m$ to $n + 1$; since $\omega \notin \Phi$, this map must have finite image, whence there are $i_0 < i_1 < \cdots$ with $x_{i_0} \ge x_{i_1} \ge \cdots$, a contradiction, which proves (i). It follows that for a Φ-continuous Φ-suplattice X with countable increasing joins, every $\twoheaddownarrow x \in \Phi(X)$ is closed under countable increasing joins. In particular, for $x_0 \ll x_1 \ll \cdots \in X$, $x := \bigvee_n x_n$ has $x_n \ll x$ for each n, whence $x \ll x$. Now for any $y \in X$ and $x_0 \ll y$, by interpolation (Proposition 2.6(c)) we may find $x_0 \ll x_1 \ll \cdots \ll y$, whence $x := \bigvee_n x_n$ is Φ-compact with $x_0 \le x \ll y$; since $y = \bigvee \twoheaddownarrow y$, it follows that X is Φ-algebraic, proving (ii). □

Corollary 2.13. *For a join doctrine Φ, the following are equivalent:*

(i) $\omega \notin \Phi$.

(ii) $\mathbb{I}$ is Φ-algebraic.

(iii) Every Φ-continuous lattice is Φ-algebraic. □

3 Commuting meets and joins

We are interested in recovering Φ-continuous lattices from their dual algebras of morphisms (to 2 or $\mathbb{I}$). In order to do so, by general duality theory, the dual algebras must be equipped with all operations which commute with the Φ-continuous lattice operations of arbitrary meets and Φ-joins. Thus, we now review the theory of classes of commuting meets and joins, again due in the general enriched categories context to [KS05], although the posets case is much simpler.

It is convenient to treat a "class of meets" as simply the order-dual of a "class of joins". Thus, given a join doctrine Φ, we will refer to $\Phi^{\mathsf{op}} := \{\phi^{\mathsf{op}} \mid \phi \in \Phi\}$ as a **meet doctrine**, and a meet indexed by $\phi^{\mathsf{op}} \in \Phi^{\mathsf{op}}$ as a **Φ^{op}-meet**. A poset with all Φ^{op}-meets is a **Φ^{op}-inflattice**, with the category of all such denoted $\Phi^{\mathsf{op}}\mathsf{Inf}$. A **$\Phi^{\mathsf{op}}$-filter** is an upper sub-Φ^{op}-inflattice. The free Φ^{op}-inflattice generated by a poset X is $\Phi(X^{\mathsf{op}})^{\mathsf{op}}$.

Definition 3.1 (see [KS05])**.** For two join doctrines Φ, Ψ, where we regard Ψ^{op} as a meet doctrine, to say that **Ψ^{op}-meets commute with Φ-joins in** 2 means that for any posets X, Y,

$$\forall \phi \in \Phi(Y)\, \forall \psi \in \Psi(X)\, \forall F : X^{\mathsf{op}} \times Y \to 2 \left(\bigwedge_{x \in \psi} \bigvee_{y \in \phi} F(x, y) = \bigvee_{y \in \phi} \bigwedge_{x \in \psi} F(x, y) \right)$$

(where F runs over monotone maps). By currying F, this is equivalent to

$$\forall \phi \in \Phi(Y)\, \forall \psi \in \Psi(X)\, \forall f : Y \to \mathcal{L}(X) \left(\psi \subseteq \bigcup_{y \in \phi} f(y) \iff \exists y \in \phi\, (\psi \subseteq f(y)) \right)$$
$$\iff \forall \psi \in \Psi(X)\, (\psi \in \mathcal{L}(X) \text{ is } \Phi\text{-compact}).$$

We write $\Phi^*(X) := \mathcal{L}(X)_\Phi$ for the Φ-compact lower sets $\psi \subseteq X$, i.e., those indexing meets commuting with Φ-joins in 2. Note that by order-duality, the roles of Φ, Ψ may be swapped. Thus

$$\Psi^{\mathsf{op}}\text{-meets commute with } \Phi\text{-joins in } 2 \iff \Psi \subseteq \Phi^* \iff \Phi \subseteq \Psi^* \quad (\text{as submonads of } \mathcal{L}).$$

Remark 3.2. The above definition of Φ^*, which follows [KS05], yields *a priori* a submonad of $\mathcal{L}$. But such a submonad automatically obeys the saturation condition $(*)$ of Remark 2.3, since given an order-embedding $i : X \hookrightarrow X'$ and poset Y, a monotone $F : X^{\mathsf{op}} \times Y \to 2$ may be extended along i to $F' : X'^{\mathsf{op}} \times Y \to 2$ (e.g., the left Kan extension $F'(x', y) := \bigvee_{x \in i^{-1}(\uparrow x')} F(x, y)$), so that for $\psi \in \mathcal{L}(X)$, the ψ^{op}-meet of F commutes with all Φ-joins iff the $i_*(\psi)^{\mathsf{op}}$-meet of F' does. Thus by Remark 2.3, we may equally well regard Φ^* as a class of posets. Namely, for a poset ψ,

$$\psi \in \Phi^* \iff \psi \in \Phi^*(\psi) = \mathcal{L}(\psi)_\Phi$$
$$\iff \text{whenever } \psi \text{ is a } \Phi\text{-union of lower subsets, one of them is } \psi.$$

Note moreover that this reasoning applies to Φ^* even if Φ is only a submonad of $\mathcal{L}$ to begin with; this justifies our claim from Remark 2.3 that for our duality-theoretic purposes, it suffices to consider "join doctrines" which are classes of posets, rather than arbitrary submonads of $\mathcal{L}$ as in [KS05].

Remark 3.3. Φ-joins commute with Ψ^{op}-meets in 2 iff they do in the unit interval $\mathbb{I}$. This follows from the facts that 2 is a complete sublattice of $\mathbb{I}$, while $\mathbb{I}$ is a complete lattice homomorphic image via $\bigvee : \mathcal{L}(\mathbb{I}) \twoheadrightarrow \mathbb{I}$ (by complete distributivity, Example 2.11) of a complete sublattice $\mathcal{L}(\mathbb{I}) \subseteq 2^{\mathbb{I}}$.

Remark 3.4. If $\phi \in \Psi^*(X)$ for a Ψ-suplattice X, then by considering the indicator function of $\leq \subseteq X^{\mathsf{op}} \times X$, we get that ϕ must be a Ψ-ideal. (The converse is false in general: for $\Psi =$ directed posets, a Ψ-ideal is a Scott-closed subset, but only finite meets commute with directed joins.)

Proposition 3.5 ([KS05, 8.9, 8.11, 8.13])**.** *Let Φ, Ψ be two join doctrines such that Ψ^{op}-meets commute with Φ-joins in 2. The following are equivalent:*

(i) For every poset X, $\mathcal{L}(X)$ is generated under Φ-joins by $\Psi(X) \subseteq \mathcal{L}(X)_\Phi$.

(ii) For every Ψ-suplattice X, $\Phi(X)$ consists precisely of all Ψ-ideals in X.

(iii) For every poset X, there is a sub-Ψ-suplattice $\Psi'(X) \subseteq \mathcal{L}(X)$ containing all principal ideals $\downarrow x$ (e.g., $\Psi'(X) = \mathcal{L}(X)$ or $\Psi'(X) = \Psi(X)$) such that $\Phi(\Psi'(X))$ contains all Ψ-ideals in $\Psi'(X)$.

If these hold, then in fact $\Psi(X) = \mathcal{L}(X)_\Phi = \Phi^*(X)$, whence $\mathcal{L}(X) \cong \Phi(\Psi(X))$ is Φ-algebraic, whence in particular Φ is a continuous join doctrine; and similarly $\Phi = \Psi^*$.

If these hold, we call Φ a **sound join doctrine, dual to the sound meet doctrine** Ψ^{op}. Thus, Φ is a sound join doctrine iff $\mathcal{L}(X) \cong \Phi(\Phi^*(X))$, iff $\Phi(X)$ contains every Φ^*-ideal in a Φ^*-suplattice X. (Warning: this notion is *not* preserved under swapping Φ, Ψ, in contrast to Definition 3.1.)

Proof. (ii) $\Longrightarrow$ (iii) is obvious.

(iii) $\Longrightarrow$ (i): For any $\theta \in \mathcal{L}(X)$, clearly $\Psi'(X) \cap {\downarrow}\theta = \{\psi \in \Psi'(X) \mid \psi \subseteq \theta\}$ is a Ψ-ideal in $\Psi'(X)$, thus by (iii) is in $\Phi(\Psi'(X))$; and its union is θ, which is thus a Φ-join of elements of $\Psi(X)$.

(i) $\Longrightarrow$ (ii): For every $\theta \in \mathcal{L}(X)$, the Ψ-ideal $\langle\theta\rangle$ it generates is in $\Phi(X)$: this is true for $\theta \in \Psi(X)$ since $\langle\theta\rangle = {\downarrow}\bigvee\theta$, and is true for a Φ-join $\theta = \bigcup_i \theta_i$ if it is true for each θ_i since $\langle\theta\rangle = \bigcup_i \langle\theta_i\rangle$ (using that Ψ^{op}-meets commute with Φ-joins in 2), thus is true for all $\theta \in \mathcal{L}(X)$ by (i). Conversely, as noted above, every $\phi \in \Phi(X)$ is a Ψ-ideal.

The last sentence follows from (i), (ii), and Remark 3.4, which imply that $\Phi(X) = \Psi^*(X)$ for a Ψ-suplattice X, hence for every poset X by applying $(*)$ in Remark 2.3 to $\downarrow : X \to \Psi(X)$. □

Lemma 3.6. *For any join doctrine Φ, we have $\omega \in \Phi$ iff $\omega \notin \Phi^*$.*

Proof. $\omega \notin \Phi \cap \Phi^*$ since ω-joins do not commute with ω^{op}-meets in 2. If $\omega \notin \Phi^*$, i.e., $\omega \in \mathcal{L}(\omega)$ is not Φ-compact, then ω is a Φ-union of proper lower subsets of ω; the order-type of this union must clearly be ω. (This argument is due to the referee; my original proof assumed soundness of Φ.) □

Corollary 3.7 (generalized Hofmann–Mislove–Stralka duality)**.** *Let Φ be a sound join doctrine, dual to the meet doctrine $\Psi^{\mathsf{op}} = \Phi^{*\mathsf{op}}$. We have a dual equivalence of categories*

$$\Phi\mathsf{AlgLat}^{\mathsf{op}} \underset{\Psi^{\mathsf{op}}\mathsf{Inf}(-,2)}{\overset{\Phi\mathsf{AlgLat}(-,2)}{\rightleftarrows}} \Psi^{\mathsf{op}}\mathsf{Inf}.$$

We may replace $\Phi\mathsf{AlgLat}$ *with* $\Phi\mathsf{CtsLat}$ *iff $\omega \notin \Phi$, i.e., $\omega \in \Psi$.*

Proof. For a Φ-algebraic lattice X, a morphism $X \to 2$ is the indicator function of $\uparrow x$ for Φ-algebraic x. For a Ψ^{op}-inflattice A, a morphism $A \to 2$ is the indicator function of a Ψ^{op}-filter. So we have

$$\Phi\mathsf{AlgLat}(X,2) \cong X_\Phi^{\mathsf{op}}, \qquad \Psi^{\mathsf{op}}\mathsf{Inf}(A,2) \cong \Phi(A^{\mathsf{op}}).$$

Now the adjunction (co)unit on the left is given by, for $X \in \Phi\mathsf{AlgLat}$, the evaluation map

$$\begin{aligned} X &\longrightarrow \Psi^{\mathsf{op}}\mathsf{Inf}(\Phi\mathsf{AlgLat}(X,2),2) \\ x &\longmapsto (f \mapsto f(x)), \end{aligned}$$

which via the above isomorphisms becomes the canonical isomorphism $X \cong \Phi(X_\Phi)$ characterizing algebraicity. Similarly, for $A \in \Psi^{\mathsf{op}}\mathsf{Inf}$, the unit $A \to \Phi\mathsf{AlgLat}(\Psi^{\mathsf{op}}\mathsf{Inf}(A,2),2)$ is the canonical isomorphism $A^{\mathsf{op}} \cong \Phi(A^{\mathsf{op}})_\Phi$. By Corollary 2.13, $\Phi\mathsf{AlgLat} = \Phi\mathsf{CtsLat}$ iff $\mathbb{I}$ is Φ-algebraic, iff $\omega \notin \Phi$. □

Example 3.8. Φ = directed posets forms a sound join doctrine, dual to Ψ^{op} = "finite meets", i.e., Ψ = the class of posets with finite cofinality. In this case, Corollary 3.7 becomes the classical Hofmann–Mislove–Stralka duality [HMS74] between (unital) meet-semilattices and algebraic lattices.

Similarly, the join doctrine Φ of κ-directed posets for an uncountable regular cardinal κ is sound, dual to κ-ary meets. But since $\omega \notin \Phi$ for uncountable κ, we get a duality between κ-meet-semilattices and κ-*continuous* lattices.

We now show that there are very few sound join doctrines $\Phi \ni \omega$, for which $\Phi\mathsf{AlgLat} \neq \Phi\mathsf{CtsLat}$: essentially, they are only the classical cases of continuous and completely distributive lattices (Examples 2.9 and 2.10), plus the minor variations including/excluding empty joins.

Theorem 3.9. *There are precisely 4 sound join doctrines $\Phi \ni \omega$, dual to Ψ^{op}:*

(i) Φ = *directed posets,* Ψ = *posets with finite cofinality;*

(ii) Φ = *empty or directed posets,* Ψ = *nonempty posets with finite cofinality;*

(iii) Φ = *nonempty posets,* Ψ = *posets which are empty or have greatest element;*

(iv) Φ = *all posets,* Ψ = *posets with greatest element.*

Proof. It is well-known and easily seen that each of these 4 cases is sound; we show the converse.

First, we show that Φ must contain every directed poset, i.e., every poset in Ψ must have finite cofinality. For every set X, Φ contains the finite powerset $\mathcal{P}_\omega(X)$, since this is a Ψ-ideal in the full powerset $\mathcal{P}(X)$, since by Proposition 2.12(i) (applied to $\Psi \not\ni \omega$), every $\psi \in \Psi(\mathcal{P}_\omega(X))$ can have neither a strictly increasing sequence nor infinitely many maximal elements, thus must be finite. Now for every join-semilattice X, we have a monotone surjection $\bigvee : \mathcal{P}_\omega(X) \twoheadrightarrow X$, whence $X \in \Phi$. Since every directed poset ϕ is cofinal in the free join-semilattice it generates, it follows that $\phi \in \Phi$.

So Ψ is determined by the finite antichains n in it. If some $n > 1$ is in Ψ, then by induction so is each $n^k \cong \bigsqcup_{i \in n} n^{k-1}$; now every $m \geq 1$ admits a surjection $n^k \twoheadrightarrow m$, whence $m \in \Psi$. □

4 $\mathbb{U}$-posets

Henceforth, we assume $\Phi \ni \omega$ is a sound join doctrine, dual to Ψ^{op}, so one of the cases in Theorem 3.9. Then Hofmann–Mislove–Stralka duality does not apply to all Φ-continuous lattices, and so we would like to formulate a duality based on morphisms to $\mathbb{I}$ instead of 2.

By Remark 3.3, the dual algebra $\Phi\mathsf{CtsLat}(X, \mathbb{I})$ will still be equipped with Ψ^{op}-meets. But these are not all the operations on $\mathbb{I}$ commuting with the Φ-continuous lattice operations: clearly any complete lattice homomorphism $\mathbb{I} \to \mathbb{I}$ does as well. We thus introduce the following notions:

Definition 4.1. Let $\mathbb{U} := \mathsf{CLat}(\mathbb{I}, \mathbb{I})$ denote the partially ordered monoid of all complete lattice homomorphisms $\mathbb{I} \to \mathbb{I}$, i.e., surjective monotone maps.

A **$\mathbb{U}$-poset** is a poset equipped with a monotone (in both variables) action of the monoid $\mathbb{U}$. Denote the category of these (and equivariant monotone maps) by $\mathbb{U}\mathsf{Pos}$.

A **$\mathbb{U}$-Ψ^{op}-inflattice** is a $\mathbb{U}$-poset which is also a Ψ^{op}-inflattice such that the action of each $u \in \mathbb{U}$ preserves Ψ^{op}-meets. Denote the category of these by $\mathbb{U}\Psi^{\mathsf{op}}\mathsf{Inf}$.

Definition 4.2. Let $\dot{+}, \dot{-}$ denote truncated $+, -$ on $\mathbb{I}$; note that they obey the adjunction

$$r \dot{-} s \leq t \iff r \leq s \dot{+} t. \tag{4.3}$$

For a $\mathbb{U}$-poset A and $a, b \in A$, define

$$a \le_r b \;:\!\iff\; \forall u, v \in \mathbb{U}\,(u((-) \dotplus r) \le v \implies u(a) \le v(b)),$$
$$\rho(a,b) := \bigwedge\{r \in \mathbb{I} \mid a \le_r b\},$$
$$d(a,b) := \rho(a,b) \vee \rho(b,a).$$

Remark 4.4. In the definition of $\le_r$, instead of testing $\forall u, v$, it is enough to test any particular $u \in \mathbb{U}$ which restricts to an order-isomorphism $u : [r,1] \cong [0,1]$ (e.g., the linear such isomorphism extended by 0 on $[0,r]$), so that $v := u((-) \dotplus r) \in \mathbb{U}$. Indeed, for any other $u', v' \in \mathbb{U}$ with $u'((-) \dotplus r) \le v'$, there is $w \in \mathbb{U}$ with $u' = w \circ u$, whence $u'(a) = w(u(a)) \le w(v(a)) \le v'(a)$.

Remark 4.5. There is an evident order-duality for $\mathbb{U}$-posets A: let $u \in \mathbb{U}$ act on the order-dual A^{op} via $1 - u(1 - (-))$; this reverses each $\le_r$, and turns ρ into $\rho^{\mathsf{op}}(a,b) := \rho(b,a)$.

Intuitively, $a \le_r b$ means "$a \le b \dotplus r$". The following properties justify this interpretation:

Proposition 4.6. *In $\mathbb{I}$, we have $a \le_r b \iff a \le b \dotplus r$, whence $\rho(a,b) = a \dot- b$ and $d(a,b) = |a-b|$.*

Proof. If $a \le b \dotplus r$, then for every $u, v \in \mathbb{U}$ with $u((-) \dotplus r) \le v$, we have $u(a) \le u(b \dotplus r) \le v(b)$.

For the converse, the case $r = 1$ is vacuous; thus we may assume $r < 1$. Note that $(-) \dotplus r : \mathbb{I} \to \mathbb{I}$ can be written as $u^\times \circ v$ where $v := 1 \wedge (-)/(1-r)$, $u := v((-) \dot- r)$, and $u^\times$ is the right adjoint of u. Now from $a \le_r b$ and $u((-) \dotplus r) = v$, we get $u(a) \le v(b)$, whence $a \le u^\times(v(b)) = b \dotplus r$. □

Lemma 4.7. *In every $\mathbb{U}$-poset A, we have the following, for $r, s, t \in \mathbb{I}$, $u, v \in \mathbb{U}$, $a, b, c \in A$:*

(a) $r \le s \;\&\; a \le_r b \implies a \le_s b$.

(b) $\le_0$ *is the same as* $\le$.

(c) $a \le_r b \le_s c \implies a \le_{r \dotplus s} c$.

(d) ρ *is a pseudoquasimetric:* $\rho(a,a) = 0$, *and* $\rho(a,b) + \rho(b,c) \ge \rho(a,c)$. *Thus,* d *is a pseudometric.*

(e) $u((-) \dotplus r) \le v \dotplus s \;\&\; a \le_r b \implies u(a) \le_s v(b)$. *Thus,* $\rho(u(a), v(a)) \le \rho(u,v) := \bigvee(u \dot- v)$, *i.e., the $\mathbb{U}$-action is 1-Lipschitz in the first variable with respect to the ℓ^∞-quasimetric on $\mathbb{U}$. Moreover, if $u \in \mathbb{U}$ is uniformly continuous with modulus $\mu : \mathbb{I} \to \mathbb{I}$, i.e., $u(r) \dot- u(s) \le \mu(r \dot- s)$, then the action of u is uniformly continuous with the same modulus: $\rho(u(a), u(b)) \le \mu(\rho(a,b))$.*

(f) $u^\times((-) \dotplus r) \le v \dotplus s \;\&\; u(a) \le_r b \implies a \le_s v(b)$ *(where $u^\times$ is the right adjoint of u).*

In a $\mathbb{U}$-Ψ^{op}-inflattice, we moreover have, for $\psi, \psi' \in \Psi(A^{\mathsf{op}})$:

(g) $a \le_r \bigwedge \psi \iff \forall b \in \psi\,(a \le_r b)$. *Thus,* $\rho(\bigwedge \psi, \bigwedge \psi') \le \bigwedge_{a \in \psi} \bigvee_{b \in \psi'} \rho(a,b)$.

Proof. (a) and (b) are straightforward, as is (d) given the previous parts.

(c) For $u, w \in \mathbb{U}$ with $u((-) \dotplus (r \dotplus s)) \le w$, we have $v := u((-) \dotplus r) \in \mathbb{U}$ with $u((-) \dotplus r) \le v$ and $v((-) \dotplus s) \le w$, whence $u(a) \le v(b) \le w(c)$.

(e) For $u', v' \in \mathbb{U}$ with $u'((-) \dotplus s) \le v'$, we have $u'(u((-) \dotplus r)) \le u'(v(-) \dotplus s) \le v' \circ v$, whence $u'(u(a)) \le v'(v(b))$. For the last assertion: $u(r) \dot- u(s) \le \mu(r \dot- s)$ means $u((-) \dotplus r) \le u(-) \dotplus \mu(r)$.

(f) The assumption is equivalent to $(-) \dot- s \le v(u(-) \dot- r)$; thus for $u', v' \in \mathbb{U}$ with $u'((-) \dotplus s) \le v'$, we have $u' \le v'((-) \dot- s) \le v'(v(u(-) \dot- r))$, whence $u'(a) \le v'(v(u(a) \dot- r)) \le v'(v(b))$.

(g) $\implies$ and the last assertion follow from (c). For $\impliedby$: for $u, v \in \mathbb{U}$ with $u((-) \dotplus r) \le v$, we have $u(a) \le \bigwedge_{b \in \psi} v(b) = v(\bigwedge \psi)$. □

For general background on (pseudo)quasimetrics, see e.g., [Kün09]. A pseudoquasimetric ρ as above induces a topology, where a basic neighborhood of $a \in A$ is $\{b \in A \mid \rho(a,b) < r\}$ for some $r > 0$. Thus the closure of $B \subseteq A$ is the set of all $a \in A$ such that

$$\rho(a, B) = \bigwedge_{b \in B} \rho(a,b) = 0,$$

which is in particular a lower set. To avoid confusion, we will call a closed set in this topology a **ρ-closed lower set**, and denote the set of all such by $\overline{\mathcal{L}}(A) \subseteq \mathcal{L}(A)$. We will also say **$\rho^{\mathsf{op}}$-closed upper set** $B \subseteq A$ for the order-dual notion, i.e., if $\rho(B, a) = 0$ then $a \in B$; the set of all such is thus $\overline{\mathcal{L}}(A^{\mathsf{op}})$. For a $\mathbb{U}$-Ψ^{op}-inflattice A, recalling that $\Phi(A^{\mathsf{op}})$ consists of Ψ^{op}-filters by soundness, let

$$\overline{\Phi}(A^{\mathsf{op}}) := \Phi(A^{\mathsf{op}}) \cap \overline{\mathcal{L}}(A^{\mathsf{op}})$$

denote the **ρ^{op}-closed Ψ^{op}-filters** in A.

Lemma 4.8. *If $\phi \in \Phi(A^{\mathsf{op}})$ is a Ψ^{op}-filter, then so is the ρ^{op}-closure $\overline{\phi}$.*

Proof. This follows from the facts that Ψ^{op} is a class of finite meets by Theorem 3.9, and that Ψ^{op}-meets are Lipschitz by Lemma 4.7(g). ☐

As usual for actions, a subset $B \subseteq A$ of a $\mathbb{U}$-poset is **$\mathbb{U}$-invariant** if it is closed under the action. For a class of sets $\Gamma(A)$, we write $\Gamma^{\mathbb{U}}(A)$ for the $\mathbb{U}$-invariant members, e.g., $\mathcal{L}^{\mathbb{U}}(A), \overline{\Phi}^{\mathbb{U}}(A)$.

Lemma 4.9. *If $\phi \in \mathcal{P}^{\mathbb{U}}(A)$ is a $\mathbb{U}$-invariant filter base, then its ρ^{op}-closure $\overline{\phi}$ is a $\mathbb{U}$-invariant Ψ^{op}-filter, hence is the $\mathbb{U}$-invariant ρ-closed Ψ^{op}-filter generated by ϕ.*

Proof. By uniform continuity of the action of each u (Lemma 4.7(e)), $\overline{\phi}$ is $\mathbb{U}$-invariant. It is also upper, since every ρ^{op}-closed set is, thus it is also the ρ^{op}-closure of the upward closure of ϕ, which is a Ψ^{op}-filter since Ψ^{op}-meets are finite by Theorem 3.9, whence so is $\overline{\rho}$ by the preceding lemma. ☐

Proposition 4.10. *For a $\mathbb{U}$-Ψ^{op}-inflattice A, we have an order-isomorphism*

$$\begin{aligned} \mathbb{U}\Psi^{\mathsf{op}}\mathsf{Inf}(A, \mathbb{I}) &\cong \overline{\Phi}^{\mathbb{U}}(A^{\mathsf{op}}) = \{\mathbb{U}\text{-}invariant\ \rho^{\mathsf{op}}\text{-}closed\ \Psi^{\mathsf{op}}\text{-}filters\ in\ A\} \\ f &\mapsto f^{-1}(1) \\ 1 - \rho(\phi, -) &\leftarrow\!\shortmid \phi. \end{aligned}$$

Proof. For ease of notation, we will prove the dual statement that for a $\mathbb{U}$-Ψ-suplattice A,

$$\begin{aligned} \mathbb{U}\Psi\mathsf{Sup}(A, \mathbb{I})^{\mathsf{op}} &\cong \overline{\Phi}^{\mathbb{U}}(A) = \{\mathbb{U}\text{-invariant } \rho\text{-closed } \Psi\text{-ideals in } A\} \\ f &\mapsto f^{-1}(0) \\ \rho(-, \phi) &\leftarrow\!\shortmid \phi. \end{aligned}$$

It is immediate from the definitions that for a $\mathbb{U}$-equivariant Ψ-join-preserving $f : A \to \mathbb{I}$, $f^{-1}(0) \subseteq A$ is $\mathbb{U}$-invariant ρ-closed lower, and also that a ρ-closed lower $\phi \subseteq A$ is equal to $\rho(-, \phi)^{-1}(0)$.

We now check that for a $\mathbb{U}$-invariant Ψ-ideal $\phi \subseteq A$, $\rho(-, \phi) : A \to \mathbb{I}$ is $\mathbb{U}$-equivariant Ψ-join-preserving (it is clearly monotone). For $\psi \in \Psi(A)$,

$$\begin{aligned} \rho(\textstyle\bigvee \psi, \phi) &= \textstyle\bigwedge_{b \in \phi} \bigvee_{a \in \psi} \rho(a,b) && \text{by the dual of Lemma 4.7(g)} \\ &= \textstyle\bigvee_{a \in \psi} \bigwedge_{b \in \phi} \rho(a,b) && \text{because } \Phi \subseteq \Psi^* \text{ (Remark 3.3)} \\ &= \textstyle\bigvee_{a \in \psi} \rho(a, \phi); \end{aligned}$$

thus $\rho(-,\phi)$ preserves Ψ-joins. To check $\mathbb{U}$-equivariance: let $u \in \mathbb{U}$ and $a \in A$. We have

$$\begin{aligned}\rho(u(a),\phi) &= \textstyle\bigwedge_{b\in\phi}\rho(u(a),b) = \bigwedge\{r \in \mathbb{I} \mid u(a) \le_r b \in \phi\},\\ u(\rho(a,\phi)) &= u(\textstyle\bigwedge_{b\in\phi}\rho(a,b)) = \bigwedge_{b\in\phi}u(\rho(a,b)) = \bigwedge\{u(r) \mid a \le_r b \in \phi\}.\end{aligned}$$

For each $a \le_r b \in \phi$, find

$$u((-) \dotplus r) \dot- u(r) \le v \in \mathbb{U},$$

whence $u(a) \le_{u(r)} v(b) \in \phi$ by Lemma 4.7(e); this proves $u(\rho(a,\phi)) \ge \rho(u(a),\phi)$. Conversely, for $u(a) \le_r b \in \phi$ with $r < 1$, let $u^\times$ be the right adjoint of u, and similarly to before, find

$$u^\times((-) \dotplus r) \dot- u^\times(r) \le v \in \mathbb{U},$$

whence $a \le_{u^\times(r)} v(b) \in \phi$ by Lemma 4.7(f), whence $u(\rho(a,\phi)) \le r$; so $\rho(u(a),\phi) \ge u(\rho(a,\phi))$.

Finally, we check that for $\mathbb{U}$-equivariant monotone $f : A \to \mathbb{I}$, we have $f = \rho(-, f^{-1}(0))$. We have $\le$ since f is 1-Lipschitz. Conversely, for $a \in A$ with $f(a) < 1$, find $(-) \dot- f(a) \le u \in \mathbb{U}$ with $u(f(a)) = 0$; then $a \le_{f(a)} u(a)$ by Lemma 4.7(e), so $\rho(a, f^{-1}(0)) \le \rho(a, u(a)) \le f(a)$. □

The $\mathbb{U}$-poset $\mathbb{I}$ obeys the following additional axioms, which must thus also hold in the dual of a Φ-continuous lattice:

Definition 4.11. We call a $\mathbb{U}$-poset A **Archimedean** if it obeys

$$\forall r > 0\,(a \le_r b) \implies a \le b.$$

We call A **(Cauchy-)complete** if it is Archimedean and also complete in the metric d.

Definition 4.12. We call a $\mathbb{U}$-poset A **unstackable** if for any $0 < r < 1$ and $u, v \in \mathbb{U}$ restricting to order-isomorphisms $u : [0,r] \cong [0,1]$ and $v : [r,1] \cong [0,1]$, we have

$$u(a) \le u(b) \;\&\; v(a) \le v(b) \implies a \le b.$$

We call A **stackable** if it is unstackable and for r, u, v as above and $a, b \in A$ such that $v'(b) \le u'(a)$ for all $u', v' \in \mathbb{U}$, there is a (unique, by unstackability) $c \in A$ with $u(c) = a$ and $v(c) = b$.

Intuitively, stackability means that, thinking of A as the dual of a Φ-continuous lattice X, we may specify $A \ni a : X \to \mathbb{I}$ via its restrictions to its sublevel and superlevel sets $a^{-1}([0,r]), a^{-1}([r,1])$.

Remark 4.13. As in Remark 4.4, it is enough to take some particular u, v above. Also, it is enough to take some particular r (e.g., $1/2$), since we may move r around via an order-isomorphism $\mathbb{I} \cong \mathbb{I}$.

Lemma 4.14. *If A is (un)stackable, then more generally, for $0 = r_0 < r_1 < \cdots < r_n = 1$ and $u_1, \ldots, u_n \in \mathbb{U}$ restricting to $u_i : [r_{i-1}, r_i] \cong [0,1]$, for $a_1, \ldots, a_n \in A$ such that $v'(a_{i+1}) \le u'(a_i)$ for all $u', v' \in \mathbb{U}$, there is (at most one, depending monotonically on $(a_1, \ldots, a_n)$) $a \in A$ with $u_i(a) = a_i$.*

Proof. By a straightforward induction on n. □

Lemma 4.15. *If A is unstackable, then more generally, for $0 \le r = r_0 < r_1 < \cdots < r_n = 1$ and $u_1, \ldots, u_n \in \mathbb{U}$ with $u_i : [r_{i-1}, r_i] \cong [0,1]$, so that $u_i((-) \dotplus r) \in \mathbb{U}$, for any $a, b \in A$, we have*

$$u_1(a) \le u_1(b \dotplus r) \;\&\; \cdots \;\&\; u_n(a) \le u_n(b \dotplus r) \implies a \le_r b.$$

Proof. By Remark 4.4, it suffices to check that for $w \in \mathbb{U}$ with $w : [r,1] \cong [0,1]$, we have $w(a) \le w(b \dotplus r)$; this follows from applying the preceding lemma to $u_i \circ w^{-1} : [w(r_{i-1}), w(r_i)] \cong [0,1]$. □

5 The duality

Let $\mathsf{CSt}\mathbb{U}\Psi^{\mathsf{op}}\mathsf{Inf} \subseteq \mathbb{U}\Psi^{\mathsf{op}}\mathsf{Inf}$ denote the full subcategory of complete stackable $\mathbb{U}$-Ψ^{op}-inflattices. Since the Φ-continuous lattice and $\mathbb{U}$-Ψ^{op}-inflattice structures on $\mathbb{I}$ commute, we have a dual adjunction

$$\Phi\mathsf{CtsLat}^{\mathsf{op}} \underset{\mathbb{U}\Psi^{\mathsf{op}}\mathsf{Inf}(-,\mathbb{I})}{\overset{\Phi\mathsf{CtsLat}(-,\mathbb{I})}{\rightleftarrows}} \mathsf{CSt}\mathbb{U}\Psi^{\mathsf{op}}\mathsf{Inf} \subseteq \mathbb{U}\Psi^{\mathsf{op}}\mathsf{Inf}. \tag{5.1}$$

Theorem 5.2. *For every Φ-continuous lattice X, the evaluation map*

$$\begin{aligned} \eta : X &\longrightarrow \mathbb{U}\Psi^{\mathsf{op}}\mathsf{Inf}(\Phi\mathsf{CtsLat}(X, \mathbb{I}), \mathbb{I}) \\ x &\longmapsto (f \mapsto f(x)), \end{aligned}$$

which is the (co)unit on the left side of the above adjunction, is an order-isomorphism.

Proof. Via Propositions 2.7 and 4.10, η corresponds to the map

$$\begin{aligned} \widetilde{\eta} : X &\longrightarrow \overline{\Phi}^{\mathbb{U}}(\ll^{\Phi}\mathsf{Sup}(\mathbb{I}, X)) \subseteq \mathcal{L}(\ll^{\Phi}\mathsf{Sup}(\mathbb{I}, X)) \\ x &\longmapsto \{f^+ \in \ll^{\Phi}\mathsf{Sup}(\mathbb{I}, X) \mid f^+(1) \le x\} \end{aligned}$$

whose left adjoint is easily seen to be

$$\begin{aligned} \widetilde{\eta}^+ : \mathcal{L}(\ll^{\Phi}\mathsf{Sup}(\mathbb{I}, X)) &\longrightarrow X \\ \phi &\longmapsto \textstyle\bigvee_{f^+ \in \phi} f^+(1). \end{aligned}$$

That $x \le \widetilde{\eta}^+(\widetilde{\eta}(x))$ is Urysohn's lemma for Φ-continuous lattices; see [G+03, IV-3.1, IV-3.32], [Joh82, VII 1.14, 3.2], [Xu95]. Since $x = \bigvee \twoheaddownarrow x$, it suffices to show that for each $y \ll x$ there is $f^+ \in \ll^{\Phi}\mathsf{Sup}(\mathbb{I}, X)$ with $y \le f^+(1) \le x$. Let $\mathbb{I}_2 \subseteq \mathbb{I}$ be the dyadic rationals, define $g : \mathbb{I}_2 \to X$ by $g(0) := y$, $g(1) := x$, and inductively using interpolation (Proposition 2.6(c)) so that $r < s \implies g(r) \ll g(s)$; then $f^+(r) := \bigvee g(\mathbb{I}_2 \cap [0, r))$ works.

Now let $\phi \in \overline{\Phi}^{\mathbb{U}}(\ll^{\Phi}\mathsf{Sup}(\mathbb{I}, X))$; we must show $\widetilde{\eta}(\widetilde{\eta}^+(\phi)) \subseteq \phi$. Since $\widetilde{\eta}$ preserves Φ-joins,

$$\widetilde{\eta}(\widetilde{\eta}^+(\phi)) = \textstyle\bigvee_{f^+ \in \phi} \widetilde{\eta}(f^+(1)).$$

For each $f^+ \in \phi$ and $g^+ \in \widetilde{\eta}(f^+(1))$, i.e., $g^+(1) \le f^+(1)$, we have $1 \le g(f^+(1))$, thus there is $g \circ f^+ \ge u \in \mathbb{U}$, whence $g \ge u \circ f$, so $g^+ \le (u \circ f)^+ \in \phi$ since ϕ is $\mathbb{U}$-invariant; thus $\widetilde{\eta}(f^+(1)) \subseteq \phi$. ☐

Theorem 5.3. *For every Archimedean unstackable $\mathbb{U}$-Ψ^{op}-inflattice A, the evaluation map*

$$\begin{aligned} \iota : A &\longrightarrow \Phi\mathsf{CtsLat}(\mathbb{U}\Psi^{\mathsf{op}}\mathsf{Inf}(A, \mathbb{I}), \mathbb{I}) \\ a &\longmapsto (f \mapsto f(a)) \end{aligned}$$

is an embedding. If A is stackable, its image is dense; thus if A is also complete, it is an isomorphism.

Proof. Via Propositions 2.7 and 4.10, ι corresponds to the map

$$\begin{aligned} \widetilde{\iota} : A &\longrightarrow \ll^{\Phi}\mathsf{Sup}(\mathbb{I}, \overline{\Phi}^{\mathbb{U}}(A^{\mathsf{op}}))^{\mathsf{op}} \\ a &\longmapsto (r \mapsto \min\{\phi \in \overline{\Phi}^{\mathbb{U}}(A^{\mathsf{op}}) \mid r \le 1 - \rho(\phi, a)\}). \end{aligned}$$

We claim that in fact, for $r > 0$, $\widetilde{\iota}(a)(r)$ is the ρ^{op}-closure $\overline{U_r(a)}$ of

$$U_r(a) := \{u(a) \mid u \in \mathbb{U} \ \& \ u(r) = 1\}.$$

$\overline{U_r(a)}$ is a $\mathbb{U}$-invariant Ψ^{op}-filter by Lemma 4.9. Each $u(a) \in U_r(a)$ is in each $\phi \in \overline{\Phi}^{\mathbb{U}}(A^{\mathsf{op}})$ with $r \le 1 - \rho(\phi, a)$: if $u(s) = 1$ for some $s < r$, we may let $b \in \phi$ with $b \le_{1-s} a$ to get $\phi \ni u(b \dot{-} (1-s)) \le u(a)$, while if there is no such s, we may write u as a limit of $u_0, u_1, \ldots$ for which there are such s, then use that ϕ is closed. And $r \le 1 - \rho(\overline{U_r(a)}, a)$: letting $(-) \dot{+} (1-r) \ge u \in \mathbb{U}$ with $u(r) = 1$, we have $U_r(a) \ni u(a) \le_{1-r} a$ by Lemma 4.7(e). This proves the claim.

Now to show that $\widetilde{\iota}$ is an order-embedding: let $\widetilde{\iota}(a) \ge \widetilde{\iota}(b) : \mathbb{I} \to \overline{\Phi}^{\mathbb{U}}(A^{\mathsf{op}})$, i.e., $\overline{U_r(a)} \supseteq U_r(b)$ for every $r > 0$; since A is Archimedean, it suffices to show $a \le_{2/n} b$ for all $n \ge 3$. For $i = 1, \ldots, n$, let

$$(*) \qquad v_i \in \mathbb{U}, \quad v_i : [(i-1)/n, i/n] \cong [0,1].$$

Then $v_i(b) \in U_{i/n}(b)$, so there is $u_i \in \mathbb{U}$ with $u_i(i/n) = 1$ such that

$$u_i(a) \le_{1/n} v_i(b).$$

Let $u', v' \in \mathbb{U}$ with $u'((-) \dot{+} 1/n) \le v'$; then for $2 \le i \le n-1$, we have $v_{i+1}(a) \le u'(u_i(a)) \le v'(v_i(b)) \le v_{i+1}(b \dot{+} 2/n)$ since $v_{i+1}(i/n) = 0$, $u'(u_i(i/n)) = 1$, $v'(v_i((i-1)/n)) = 0$, and $v_{i+1}((i-1)/n \dot{+} 2/n) = 1$. Thus since A is unstackable, by Lemma 4.15 we have $a \le_{2/n} b$.

Finally, suppose A is stackable, and let $f^+ \in \ll^{\Phi}\mathsf{Sup}(\mathbb{I}, \overline{\Phi}^{\mathbb{U}}(A^{\mathsf{op}}))$, left adjoint to f; we will find, for every $n \ge 2$, some $a \in A$ with $d(\iota(a), f) \le 2/n$. For $i = 1, \ldots, n$, we have $f^+((i-1)/n) \ll f^+(i/n) = \bigvee_{a \in f^+(i/n)} \overline{U_1(a)} = \bigvee_{a \in f^+(i/n)} \bigvee_{r<1} \overline{U_r(a)}$ (again by Lemma 4.9), whence

$$f^+((i-1)/n) \subseteq \overline{U_{r_i}(a_i)}$$

for some $a_i \in f^+(i/n)$ and $r_i < 1$. Let $u_i \in \mathbb{U}$ with $u_i(r_i) = 0$, and let v_i as in $(*)$. Then for $u' \in \mathbb{U}$,

$$f^+((i-1)/n) \subseteq \uparrow u'(u_i(a_i)) \subseteq \overline{U_1(u_i(a_i))},$$

since for $b \in f^+((i-1)/n) \subseteq \overline{U_{r_i}(a_i)}$, for every $s > 0$, there is $u'' \in \mathbb{U}$ with $u''(r_i) = 1$, whence $u' \circ u_i \le u''$, such that $u'(u_i(a_i)) \le u''(a_i) \le_s b$, whence $u'(u_i(a_i)) \le b$ since A is Archimedean. In particular, this holds for $b = v'(u_{i-1}(a_{i-1}))$ for every $v' \in \mathbb{U}$, so by Lemma 4.14, there is $a \in A$ with

$$v_i(a) = u_i(a_i)$$

for each i. Then

$$U_{i/n}(a) = U_1(v_i(a)) = U_1(u_i(a_i)),$$

since every $u \in \mathbb{U}$ with $u(i/n) = 1$ is $\ge u' \circ v_i$ for some $u' \in \mathbb{U}$. We now show that $d(f, \iota(a)) \le 2/n$, in terms of the left adjoints $f^+, \widetilde{\iota}(a)$: for each $t \in \mathbb{I}$, letting $1 \le i \le n$ with $t \le i/n \le t \dot{+} 1/n$,

$$\begin{aligned} \widetilde{\iota}(a)(t) = \overline{U_t(a)} \subseteq \overline{U_{i/n}(a)} = \overline{U_1(u_i(a_i))} \subseteq f^+(i/n) \subseteq f^+(t \dot{+} 1/n), \\ f^+(t \dot{-} 1/n) \subseteq f^+((i-1)/n) \subseteq \overline{U_1(u_i(a_i))} = \overline{U_{i/n}(a)} \subseteq \overline{U_{t \dot{+} 1/n}(a)} = \widetilde{\iota}(a)(t \dot{+} 1/n). \end{aligned} \qquad \square$$

Theorem 5.4. *The dual adjunction (5.1) is a dual equivalence of categories between Φ-continuous lattices and complete stackable $\mathbb{U}$-Ψ^{op}-inflattices.* $\square$

It is worth explicitly restating the duality for the two main examples of Φ:

Corollary 5.5. *Hom into $\mathbb{I}$ yields a dual equivalence of categories between completely distributive lattices and complete stackable $\mathbb{U}$-posets.* ☐

Let us say that a **$\mathbb{U}$-meet-semilattice** is a $\mathbb{U}$-poset with finite meets preserved by the $\mathbb{U}$-action.

Corollary 5.6. *Hom into $\mathbb{I}$ yields a dual equivalence of categories between continuous lattices and complete stackable $\mathbb{U}$-meet-semilattices.* ☐

We end by showing that in the presence of meets, stackability admits a simpler formulation:

Definition 5.7. Let $\widehat{\mathbb{U}} := \mathsf{CtsLat}(\mathbb{I}, \mathbb{I}) \supseteq \mathbb{U}$ be the monoid of continuous lattice morphisms $\mathbb{I} \to \mathbb{I}$, i.e., continuous monotone maps preserving 1, but possibly not 0.

A **$\widehat{\mathbb{U}}$-module** is a (unital) meet-semilattice with a $\widehat{\mathbb{U}}$-action preserving finite meets on both sides.

Proposition 5.8. *The forgetful functor is an isomorphism of categories between complete $\widehat{\mathbb{U}}$-modules and complete stackable $\mathbb{U}$-meet-semilattices. The $\le_r$ relations in a $\widehat{\mathbb{U}}$-module are given by*

$$\rho(a, b) \le r \iff a \le_r b \iff a \le b \dotplus r.$$

Proof. The characterization of $\le_r$ is proved as in Proposition 4.6.

Next, an Archimedean $\widehat{\mathbb{U}}$-module A is unstackable as a $\mathbb{U}$-poset: by Remark 4.13, it suffices to check that for $0 < r < 1$, $u := 1 \wedge (-)/r$, and $v := ((-) \dot{-} r)/(1 - r)$, if $u(a) \le u(b)$ and $v(a) \le v(b)$, then $a \le b$. Let $s > 0$, and let $r(-) \le w \in \mathbb{U}$ with equality on $[0, 1 - s]$. Then $1_{\mathbb{I}} \le (w \circ u) \wedge (v^\times \circ v) \le (-) \dotplus rs$, whence from $u(a) \le u(b)$ and $v(a) \le v(b)$ we have $a \le b \dotplus rs$, i.e., $a \le_{rs} b$ by the above. Since A is Archimedean, it follows that $a \le b$.

If moreover A is a complete $\widehat{\mathbb{U}}$-module, then it is stackable: for $a, b \in A$ such that $v'(b) \le u'(a)$ for all $u', v' \in \mathbb{U}$, with the same s, u, v, w as above, letting $c_s := w(a) \wedge v^\times(b)$, we have $u(c_s) = u(w(a)) \wedge u(v^\times(b)) = u(w(a))$ which is within distance s of a since $1_{\mathbb{U}} \le u \circ w \le (-) \dotplus s$, and $v(c_s) = v(w(a)) \wedge v(v^\times(b)) = v(v^\times(b)) = b$. In particular, by unstackability (using Lemma 4.15 and uniform continuity of u), the c_s form a Cauchy net as $s \searrow 0$, hence converge to some c such that $u(c) = a$ and $v(c) = b$. Thus the forgetful functor restricts to the claimed subcategories.

The forgetful functor is full on Archimedean $\widehat{\mathbb{U}}$-modules: the action by $w \in \widehat{\mathbb{U}} \setminus \mathbb{U}$ can be recovered from the $\mathbb{U}$-action, since $w(a) = \top$ for $w(0) = 1$, while for $0 < w(0) < 1$, by unstackability, $w(a)$ is the unique element such that $u(w(a)) = \top$ and $v(w(a)) = (v \circ w)(a)$ where u, v are as above for $r := w(0)$. Thus $\mathbb{U}$-equivariance implies $\widehat{\mathbb{U}}$-equivariance.

Conversely, in a complete stackable $\mathbb{U}$-meet-semilattice A, we may extend the $\mathbb{U}$-action to a $\widehat{\mathbb{U}}$-action by defining $w(a)$ for $0 < w(0) < 1$ to be the unique element as above.

The $\mathbb{U}$-action on an Archimedean stackable $\mathbb{U}$-poset A preserves binary meets in $\mathbb{U}$: for piecewise linear $u, v \in \mathbb{U}$, we may show $(u \wedge v)(a) = u(a) \wedge v(a)$ by unstacking over a finite partition of $[0, 1]$ on each piece of which u, v are comparable; for arbitrary u, v, take piecewise linear approximations.

Finally, on a complete stackable $\mathbb{U}$-meet-semilattice, the extended $\widehat{\mathbb{U}}$-action from above also preserves binary meets in $\widehat{\mathbb{U}}$, by a routine unstacking over $0 < w(0) < 1$. ☐

Corollary 5.9 (of Corollary 5.6 and Proposition 5.8)**.** *Hom into $\mathbb{I}$ yields a dual equivalence of categories between continuous lattices and complete $\widehat{\mathbb{U}}$-modules.* ☐

We end by noting that we currently do not know whether complete $\widehat{\mathbb{U}}$-modules can be equationally axiomatized, perhaps along the lines of [Abb19], thereby showing that $\mathsf{CtsLat}^{\mathsf{op}}$ is a variety.

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Department of Mathematics
University of Michigan
Ann Arbor, MI 48109, USA
Email: `ruiyuan@umich.edu`