On the canonical Dicks–Dunwoody structure tree

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Let $G \subseteq X^2$ be a connected simple undirected graph. By an (oriented edge) **cut**, we mean a partition of the vertex set $X = H \sqcup \neg H$, which we may identify with the first half $H \subseteq X$, such that the edge boundary between them $\delta H := G \cap (H \times \neg H)$ is finite. The collection of all cuts forms a Boolean algebra $\mathcal{H}_{\delta<\infty} = \mathcal{H}^G_{\delta<\infty}(X) \subseteq 2^X$. Two cuts $H, K \in \mathcal{H}_{\delta<\infty}$ are **nested** if one of $H, \neg H$ is disjoint from one of $K, \neg K$.

In [DD89], Dicks and Dunwoody showed that for any connected graph (X, G), there exists a "canonical" nested family of cuts $\mathcal{H}_{\asymp} \subseteq \mathcal{H}_{\delta<\infty}$ generating all of $\mathcal{H}_{\delta<\infty}$ under finite Boolean combinations. Indeed, more is true: for each $n \in \mathbb{N}$, every cut with edge boundary of size $\leq n$ is a finite Boolean combination of such cuts which are in $\mathcal{H}_{\asymp}$. The significance of nested families of cuts lies in a Stone-type duality with their trees of "ultrafilters", sometimes called *structure trees*, that forms part of the machinery around Stallings' theorem on ends of groups. The Dicks–Dunwoody result has seen numerous applications and generalizations, including in recovering Stallings' theorem and strengthenings thereof; see e.g., [Rol98], [DW13], [DK15], [Ham18].

The object of this note is to give a self-contained exposition of a version of the Dicks–Dunwoody construction. Our main goal is to clarify the precise sense in which the construction is "canonical". The construction as written in [DD89] produces a nested family which is "canonical" insofar as it is invariant under all automorphisms of the graph G; however, it is arguably "non-canonical" in that it appeals to Zorn's lemma (albeit in an automorphism-invariant way). Another way to say it is that the construction does not work in a uniform way across *all* graphs G. A different version of the construction given by Dicks [Dic18] is canonical in this stronger sense, avoiding Zorn's lemma, but only for quasi-transitive graphs (it is based on a well-ordering defined by Bergman [Ber68], which does extend to all graphs, but again depending on a well-ordering of the automorphism orbits). Finally, Dunwoody [Dun17] gave a construction which is fully canonical and works in all graphs (indeed in all networks).

We prove the following version of the Dicks–Dunwoody result, which is based on a simplified version of Dunwoody's construction, and formalizes the sense in which it is "canonical", via definability in the countably infinitary logic $\mathcal{L}_{\omega_1\omega}$ (see e.g., [Mar16]):

Theorem 1 (Dicks–Dunwoody)**.** For any connected graph (X, G), we may define a canonical nested family of connected and coconnected cuts $\mathcal{H}_{\asymp} \subseteq \mathcal{H}_{\delta<\infty}$, such that for each $n \in \mathbb{N}$, every cut with boundary of size $\leq n$ is a finite Boolean combination of such cuts which are in $\mathcal{H}_{\asymp}$; and the boundaries of such cuts in $\mathcal{H}_{\asymp}$ are defined by an $\mathcal{L}_{\omega_1\omega}$ formula $\phi_n((x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n))$ in the language of graphs, not depending on the graph (X, G).

In fact, every ingredient used in our construction appeared already in [DD89] in some form; however, we apply them in a different order, that is inspired by the approach to Stallings' theorem due to Krön [Krö10] (see also [Tse20] for an exposition of this proof).

1 Nestedness and corners

It is useful to formulate the basic properties of nestedness in the context of an abstract Boolean algebra $A = (A, \wedge, \vee, \top, \bot, \neg)$. For $a, b \in A$, their **corners** are

$$a \boxplus b := \{a \wedge b, a \wedge \neg b, \neg a \wedge b, \neg a \wedge \neg b\}.$$

We call $a \wedge b, \neg a \wedge \neg b$ **opposite corners**; same for $a \wedge \neg b, \neg a \wedge b$. We call a, b **nested** if $\bot \in a \boxplus b$, i.e., one of $a, \neg a$ is disjoint from one of $b, \neg b$; we denote this by

$$a \asymp b \ :\Longleftrightarrow\ \bot \in a \boxplus b.$$

We write

$$a^{\asymp} := \{b \in A \mid a \asymp b\}.$$

For a subset $B \subseteq A$, we write $B^{\asymp} := \bigcap_{b \in B} b^{\asymp}$. We call B **nested** if its elements are pairwise nested.

Lemma 2 (see [DD89, proof of 2.9], [Krö10, 3.1], [Tse20, 3.14], [Dun17, 2.8])**.** For any $a, b, c \in A$, if $a \asymp c$ and $b \asymp c$, then either $a \wedge b \asymp c$ or $a \vee b = \top$ (so $a \asymp b$). In other words,

$$a \vee b \neq \top \implies a^{\asymp} \cap b^{\asymp} \subseteq (a \wedge b)^{\asymp}.$$

Hence

$$a \not\asymp b \implies a^{\asymp} \cap b^{\asymp} \subseteq (a \boxplus b)^{\asymp}.$$

Proof. If one of a, b is disjoint from one of $c, \neg c$, then clearly so is $a \wedge b$, so $a \wedge b \asymp c$. Otherwise, each of $\neg a, \neg b$ is disjoint from one of $c, \neg c$. If both are disjoint from c or from $\neg c$, then so is $\neg(a \wedge b) = \neg a \vee \neg b$, so $a \wedge b \asymp c$; otherwise, $a \vee b = \top$. The last claim follows by negating a, b. $\square$

Remark 3 (see [Krö10, 3.1], [Tse20, 3.15])**.** For any $a, b \in A$, clearly

$$a^{\asymp}, b^{\asymp} \subseteq (a \wedge b)^{\asymp} \cup (\neg a \wedge \neg b)^{\asymp}$$

(e.g., disjointness from a implies disjointness from $a \wedge b$). Hence for any opposite corners c, d of a, b,

$$a^{\asymp} \cup b^{\asymp} \subseteq c^{\asymp} \cup d^{\asymp}.$$

2 Subfamilies of cuts

Now let (X, G) be a connected graph, where $G \subseteq X^2$ is the edge set. For $A, B \subseteq X$, define

$$\delta(A, B) := G \cap (A \times B), \qquad \delta A := \delta(A, \neg A);$$

thus δA is the outgoing edge boundary of A. We have $\delta A = \varnothing$ iff A is **trivial**, i.e., $\varnothing$ or X.

We define the following subsets of 2^X (denoted by decorations of $\mathcal{H}$ for "half-space"):

- $\mathcal{H}_{\delta<\infty} = \mathcal{H}^G_{\delta<\infty}(X) := \{H \subseteq X \mid |\delta H| < \infty\}$, the Boolean algebra of **cuts**.
- $\mathcal{H}_{\delta \le n} := \{H \subseteq X \mid |\delta H| \le n\}$ for each $n \in \mathbb{N}$.
- $\mathcal{H}_{\mathsf{conn}} := \{H \subseteq X \mid H, \neg H \text{ are connected (or empty)}\}$.

Lemma 4 (see [DD89, 2.7], [Krö10, 2.1], [Tse20, 3.8], [CPTT25, 5.4])**.** For any $n \in \mathbb{N}$ and $x, y \in X$, there are only finitely many $H \in \mathcal{H}_{\delta \le n} \cap \mathcal{H}_{\mathsf{conn}}$ separating x, y, i.e., such that $x \in H \not\ni y$.

Proof. Clearly, $\mathcal{H}_{\delta \le n} \subseteq 2^X$ is closed in the product topology; and for any $H \in \mathcal{H}_{\delta \le n}$, the clopen neighborhood $\{K \in 2^X \mid \delta H \subseteq \delta K\}$ isolates H from all $H \neq K \in \mathcal{H}_{\mathsf{conn}}$. Thus $\mathcal{H}_{\delta \le n} \cap \mathcal{H}_{\mathsf{conn}} \subseteq \mathcal{H}_{\delta \le n}$ is a closed subset, in which every nontrivial point $H \neq \varnothing, X$ is isolated; and so the set of $x \in H \not\ni y$ is a compact set of isolated points, hence finite. □

Corollary 5 (see [DD89, 2.8], [Tse20, 3.11], [CPTT25, 5.9])**.** For any $n \in \mathbb{N}$ and $H \in \mathcal{H}_{\delta < \infty}$, there are only finitely many $K \in \mathcal{H}_{\delta \le n} \cap \mathcal{H}_{\mathsf{conn}}$ which are non-nested with H.

Proof. If H, K are non-nested, then K separates two boundary vertices of H. □

Lemma 6 (see [DD89, proof of 2.4], [Krö10, 2.2], [Tse20, 3.17], [Dun17, 2.7])**.** Suppose $H, K \in \mathcal{H}_{\delta < \infty}$ have a pair of opposite corners whose boundaries have sizes at least those of H, K respectively. Then these corners have boundaries of the same sizes as those of H, K respectively.

Proof. By negating/swapping H, K if necessary, we may assume that

$$|\delta(H \cap K)| \ge |\delta H|, \qquad\qquad |\delta(\neg H \cap \neg K)| \ge |\delta K|.$$

But also (see Figure 7)

$$\begin{aligned}
|\delta H| + |\delta K| &= |\delta(H, \neg H)| + |\delta(K, \neg K)| \\
&\ge \big(|\delta(H \cap K, \neg H \cap K)| + |\delta(H \cap K, \neg H \cap \neg K)| + |\delta(H \cap \neg K, \neg H \cap \neg K)|\big) \\
&\quad + \big(|\delta(H \cap K, H \cap \neg K)| + |\delta(H \cap K, \neg H \cap \neg K)| + |\delta(\neg H \cap K, \neg H \cap \neg K)|\big) \\
&= \big(|\delta(H \cap K, \neg H \cap K)| + |\delta(H \cap K, \neg H \cap \neg K)| + |\delta(H \cap K, H \cap \neg K)|\big) \\
&\quad + \big(|\delta(H \cap \neg K, \neg H \cap \neg K)| + |\delta(H \cap K, \neg H \cap \neg K)| + |\delta(\neg H \cap K, \neg H \cap \neg K)|\big) \\
&= |\delta(H \cap K)| + |\delta(\neg H \cap \neg K)|.
\end{aligned}$$

Thus both of the above inequalities are equalities. □

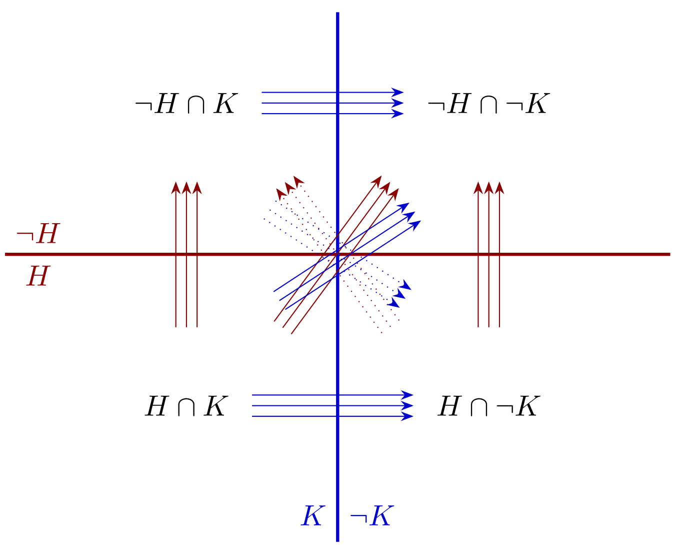

Figure 7: Counting edges between corners of a pair of cuts.

3 Irreducible cuts

For any cut $H \in \mathcal{H}_{\delta<\infty}$, using Corollary 5, define the **rank** of H to be

$$\rho(H) := (|\delta H|, |\varepsilon H|) \in \mathbb{N}^2$$

where

$$\varepsilon H := \mathcal{H}_{\delta \le |\delta H|} \cap \mathcal{H}_{\mathsf{conn}} \cap (\mathcal{H}_{\delta<|\delta H|} \cap \mathcal{H}_{\mathsf{irr}})^{\asymp} \setminus H^{\asymp}.$$

We order ranks in $\mathbb{N}^2$ lexicographically. Call $H \in \mathcal{H}_{\delta<\infty}$ **reducible** if either H or $\neg H$ is a union of two cuts of strictly smaller rank, **irreducible** otherwise. Put

- $\mathcal{H}_{\rho\le(n,k)} := \{H \in \mathcal{H}_{\delta<\infty} \mid \rho(H) \le (n,k)\}$ for each $(n,k) \in \mathbb{N}^2$.
- $\mathcal{H}_{\mathsf{irr}} := \{H \in \mathcal{H}_{\delta<\infty} \mid H \text{ is irreducible}\}$.

(Thus, the notions of rank and irreducibility are defined simultaneously by induction on $|\delta H|$.)

It is easily seen that $\rho(H) = \rho(\neg H)$, and $H \in \mathcal{H}_{\mathsf{irr}} \iff \neg H \in \mathcal{H}_{\mathsf{irr}}$. Hence, $\mathcal{H}_{\mathsf{irr}} \subseteq \mathcal{H}_{\mathsf{conn}}$, since if H has ≥ 2 components then they have strictly smaller boundary. Note also that

$$\langle \mathcal{H}_{\rho\le(n,k)} \rangle = \langle \mathcal{H}_{\rho\le(n,k)} \cap \mathcal{H}_{\mathsf{irr}} \rangle,$$

where $\langle - \rangle$ denotes the generated Boolean algebra, by an easy induction on (n,k). Thus

$$\langle \mathcal{H}_{\delta\le n} \rangle = \Uparrow\!\bigcup_k \langle \mathcal{H}_{\rho\le(n,k)} \rangle = \Uparrow\!\bigcup_k \langle \mathcal{H}_{\rho\le(n,k)} \cap \mathcal{H}_{\mathsf{irr}} \rangle = \langle \mathcal{H}_{\delta\le n} \cap \mathcal{H}_{\mathsf{irr}} \rangle. \tag{8}$$

Theorem 9. $\mathcal{H}_{\mathsf{irr}}$ is nested, hence $\mathcal{H}_{\asymp} := \mathcal{H}_{\mathsf{irr}}$ forms the desired family of cuts in Theorem 1.

Proof. We show that any $H, K \in \mathcal{H}_{\mathsf{irr}}$ are nested, by induction on $\max(|\delta H|, |\delta K|)$. Negating H and/or K if necessary, we may assume $H \cap K$ is a corner of H, K of minimal rank. Then the opposite corners $H \setminus K, K \setminus H$ have rank $\ge \rho(H \cap K)$, whence by irreducibility of H, K,

$$\rho(H \setminus K) \ge \rho(H), \qquad \rho(K \setminus H) \ge \rho(K). \tag{$*$}$$

In particular, we have the corresponding inequalities for $|\delta(-)|$. By Lemma 6, it follows that

$$|\delta(H \setminus K)| = |\delta H|, \qquad |\delta(K \setminus H)| = |\delta K|.$$

Hence by $(*)$,

$$|\varepsilon(H \setminus K)| \ge |\varepsilon H|, \qquad |\varepsilon(K \setminus H)| \ge |\varepsilon K|. \tag{$\dagger$}$$

Suppose first that $|\delta H| \ne |\delta K|$, without loss of generality $|\delta H| > |\delta K|$. If $H \not\asymp K$, then

$$\begin{aligned} \varepsilon(H \setminus K) &= \mathcal{H}_{\delta\le|\delta H|} \cap \mathcal{H}_{\mathsf{conn}} \cap (\mathcal{H}_{\delta<|\delta H|} \cap \mathcal{H}_{\mathsf{irr}})^{\asymp} \setminus (H \setminus K)^{\asymp} \\ &\subsetneqq \varepsilon H = \mathcal{H}_{\delta\le|\delta H|} \cap \mathcal{H}_{\mathsf{conn}} \cap (\mathcal{H}_{\delta<|\delta H|} \cap \mathcal{H}_{\mathsf{irr}})^{\asymp} \setminus H^{\asymp} \end{aligned}$$

by Lemma 2, the fact that every cut in $\varepsilon(H \setminus K)$ is nested with $K \in \mathcal{H}_{\delta<|\delta H|} \cap \mathcal{H}_{\mathsf{irr}}$, and that $K \in (\mathcal{H}_{\delta<|\delta H|} \cap \mathcal{H}_{\mathsf{irr}})^{\asymp} \cap (H \setminus K)^{\asymp} \setminus H^{\asymp}$ by the induction hypothesis, contradicting $(\dagger)$.

Now suppose that $|\delta H| = |\delta K|$, but $H \not\asymp K$. By Lemma 2 as above, every cut in $\varepsilon(H \setminus K)$ is non-nested with H *or* with K, and likewise for every cut in $\varepsilon(K \setminus H)$; hence

$$\varepsilon(H \setminus K) \cup \varepsilon(K \setminus H) \subsetneqq \varepsilon H \cup \varepsilon K,$$

with the inclusion being strict again because $K \in (\mathcal{H}_{\delta<|\delta H|} \cap \mathcal{H}_{\mathsf{irr}})^{\asymp} \cap (H \setminus K)^{\asymp} \setminus H^{\asymp}$, here using the previous case (instead of the induction hypothesis). But also by Remark 3,

$$\varepsilon(H \setminus K) \cap \varepsilon(K \setminus H) \subseteq \varepsilon H \cap \varepsilon K.$$

Taking cardinalities and adding yields $|\varepsilon(H \setminus K)| + |\varepsilon(K \setminus H)| < |\varepsilon H| + |\varepsilon K|$, contradicting $(\dagger)$. $\square$

Remark 10. We may "relativize" the above construction to any Boolean subalgebra $\mathcal{A} \subseteq \mathcal{H}_{\delta<\infty}$ with the property that whenever $H \in \mathcal{A}$, then every connected component of H is in $\mathcal{A}$. The definitions of rank $\rho_\mathcal{A}$, $\varepsilon_\mathcal{A}$, and irreducible cuts $\mathcal{A}_{\mathsf{irr}} \subseteq \mathcal{A}$ are the same as above, but considering only cuts in $\mathcal{A}$; the assumption on $\mathcal{A}$ guarantees that $\mathcal{A}_{\mathsf{irr}} \subseteq \mathcal{A}_{\mathsf{conn}} := \mathcal{A} \cap \mathcal{H}_{\mathsf{conn}}$ as above.

For example, let $\mathcal{C}$ be any family of connected subsets of X, or more generally closed connected subsets of the end compactification $\widehat{X} = \widehat{X}^G$ (i.e., the Stone space of $\mathcal{H}_{\delta<\infty}$, where a subset $C \subseteq \widehat{X}^G$ is *connected* if every clopen set $\widehat{H} \subseteq \widehat{X}$ for $H \in \mathcal{H}_{\delta<\infty}$ either contains C, or is disjoint from C, or has a boundary edge between two vertices in C). Then the family $\mathcal{A}_\mathcal{C} \subseteq \mathcal{H}_{\delta<\infty}$ of cuts that either contain or are disjoint from each element of $\mathcal{C}$ forms a Boolean algebra with the above property. Thus, we get a canonical nested subfamily generating $\mathcal{A}_\mathcal{C}$. This applies for instance to $\mathcal{C} = \{C, D\}$ for two disjoint closed connected sets $C, D \subseteq \widehat{X}$ (e.g., C = a geodesic between two ends, D = a third end), yielding a canonical nested generating family of cuts separating C, D.

4 $\mathcal{L}_{\omega_1\omega}$-definability of cuts

To finish, we should verify that the construction of $\mathcal{H}_{\mathsf{irr}}$ is indeed definable in the countably infinitary logic $\mathcal{L}_{\omega_1\omega}$, i.e., using countable Boolean connectives $\bigwedge, \bigvee, \neg$, as well as finitary quantifiers $\exists, \forall$. This is essentially a routine coding exercise; we will sketch the details for the sake of completeness.

The idea is to encode a nontrivial cut $\varnothing, X \neq H \in \mathcal{H}_{\delta<\infty}$ as its boundary δH, which is a finite set of pairs of vertices. For each $1 \le n \in \mathbb{N}$, define the formulas

$$\begin{aligned}\phi_{\in\delta\le n}(z, (x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)) &:= \text{"there is a path from } z \text{ to some } x_i \text{ not passing} \\ &\qquad \text{through any edge } (x_j, y_j) \text{ in either direction"}, \\ \phi_{\delta\le n}((x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)) &:= (x_1 \mathrel{G} y_1) \wedge \dots \wedge (x_n \mathrel{G} y_n) \wedge \\ &\qquad \forall z \begin{pmatrix} \neg\phi_{\in\delta\le n}(z, (x_1, y_1), \dots, (x_n, y_n)) \\ \leftrightarrow \phi_{\in\delta\le n}(z, (y_1, x_1), \dots, (y_n, x_n)) \end{pmatrix}.\end{aligned}$$

Lemma 11. A graph (X, G) satisfies $\phi_{\delta\le n}((x_1, y_1), \dots, (x_n, y_n))$ for an n-tuple of pairs of vertices iff $\{(x_1, y_1), \dots, (x_n, y_n)\} = \delta H$ for some nontrivial $H \in \mathcal{H}_{\delta\le n}$, namely

$$H_{(x_1,y_1),\dots,(x_n,y_n)} := \{z \mid \phi_{\in\delta\le n}(z, (x_1, y_1), \dots, (x_n, y_n))\}.$$

Proof. It is easily seen that if $\{(x_1, y_1), \dots, (x_n, y_n)\} = \delta H$, then $\phi_{\in\delta\le n}(z, (x_1, y_1), \dots, (x_n, y_n))$ defines precisely the elements of H; thus $\phi_{\delta\le n}((x_1, y_1), \dots, (x_n, y_n))$ holds. Conversely, suppose $\phi_{\delta\le n}((x_1, y_1), \dots, (x_n, y_n))$ holds. Then letting $H = H_{(x_1,y_1),\dots,(x_n,y_n)}$ as above, we have $x_i \in H \not\ni y_i$ for each i as witnessed by paths of length 0; thus $\{(x_1, y_1), \dots, (x_n, y_n)\} \subseteq \delta H$. If there were some $(x, y) \in \delta H$ not among the (x_i, y_i), then x would admit both a path to some x_i not passing through any (x_j, y_j), and a path through y to some y_i not passing through any (x_j, y_j), a contradiction. $\square$

From this, it is easy to define the inclusion ordering among cuts:

$$\begin{aligned}&H_{(x_1,y_1),\dots,(x_m,y_m)} \subseteq H_{(x'_1,y'_1),\dots,(x'_n,y'_n)} \\ &\qquad \iff \forall z\, (z \in H_{(x_1,y_1),\dots,(x_m,y_m)} \Rightarrow z \in H_{(x'_1,y'_1),\dots,(x'_n,y'_n)}) \\ &\qquad \iff \forall z\, (\phi_{\in\delta\le m}(z, (x_1, y_1), \dots, (x_m, y_m)) \Rightarrow \phi_{\in\delta\le n}(z, (x'_1, y'_1), \dots, (x'_n, y'_n))).\end{aligned}$$

It follows that we may define the equivalence relation of two tuples of edges representing the same cut, namely if they are $\subseteq$ each other. It is also straightforward to define (the graph of) the complement operation $\neg$ on cuts in terms of their boundaries, by just flipping edges. The lattice operations $\cap, \cup$ are also first-order definable in terms of $\subseteq$. Thus, we have essentially encoded the Boolean algebra of cuts $\mathcal{H}_{\delta<\infty}$ in $\mathcal{L}_{\omega_1\omega}$, and so we may freely refer to cuts in building more complicated formulas. (Formally speaking, we have defined an *interpretation*, in $\mathcal{L}_{\omega_1\omega}$, of the theory of Boolean algebras into the theory of connected graphs; see [Hod93, Ch. 7], [Che25, §9].)

The following properties of cuts are now straightforward to define in succession:

$$\begin{aligned}
|\delta H_{(x_1,y_1),\dots,(x_m,y_m)}| \le n &\iff \exists x_1', y_1', \dots, x_n', y_n' \left(H_{(x_1,y_1),\dots,(x_m,y_m)} = H_{(x_1',y_1'),\dots,(x_n',y_n')}\right), \\
|\delta H_{(x_1,y_1),\dots,(x_m,y_m)}| = n &\iff (|\delta H_{(x_1,y_1),\dots,(x_m,y_m)}| \le n) \wedge \neg(|\delta H_{(x_1,y_1),\dots,(x_m,y_m)}| \le n-1), \\
H \in \mathcal{H}_{\mathsf{conn}} &\iff \forall z, z' \left((z \in H \Leftrightarrow z' \in H) \Rightarrow \exists \text{ path } z \leftrightsquigarrow z' \text{ on same side of } H\right), \\
H \asymp K &\iff (H \cap K = \varnothing \text{ or } H \cap \neg K = \varnothing \text{ or } \neg H \cap K = \varnothing \text{ or } \neg H \cap \neg K = \varnothing),
\end{aligned}$$

followed by, inductively for each n, the properties

$$\text{“}|\delta H| = n \text{ and } |\varepsilon H| = k\text{”}, \\ \text{“}|\delta H| = n \text{ and } H \in \mathcal{H}_{\mathsf{irr}}\text{”}.$$

The formulas ϕ_n defining irreducible cuts with boundary of size $\le n$ desired in Theorem 1 are given by $\phi_n((x_1,y_1),\dots,(x_n,y_n)) :=$ “$H_{(x_1,y_1),\dots,(x_n,y_n)} \in \mathcal{H}_{\mathsf{irr}}$”.

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References

[Ber68] George M. Bergman, *On groups acting on locally finite graphs*, Ann. of Math. (2) **88** (1968), 335–340.

[Che25] Ruiyuan Chen, *Étale structures and the Joyal–Tierney representation theorem in countable model theory*, Bull. Symb. Log. (2025), 1–43, to appear.

[CPTT25] Ruiyuan Chen, Antoine Poulin, Ran Tao, and Anush Tserunyan, *Tree-like graphings, wallings, and median graphings of equivalence relations*, Forum Math. Sigma **13** (2025), Paper No. e64, 37.

[DD89] Warren Dicks and M. J. Dunwoody, *Groups acting on graphs*, Cambridge Studies in Advanced Mathematics, vol. 17, Cambridge University Press, Cambridge, 1989.

[Dic18] Warren Dicks, *Groups acting on graphs*, preprint (2018).

[DK15] M. J. Dunwoody and B. Krön, *Vertex cuts*, J. Graph Theory **80** (2015), no. 2, 136–171.

[Dun17] Martin J. Dunwoody, *Structure trees, networks and almost invariant sets*, Groups, graphs and random walks, London Math. Soc. Lecture Note Ser., vol. 436, Cambridge Univ. Press, Cambridge, 2017, pp. 137–175.

[DW13] Volker Diekert and Armin Weiß, *Context-free groups and their structure trees*, Internat. J. Algebra Comput. **23** (2013), no. 3, 611–642.

[Ham18] Matthias Hamann, *Accessibility in transitive graphs*, Combinatorica **38** (2018), no. 4, 847–859.

[Hod93] Wilfrid Hodges, *Model theory*, Encyclopedia of Mathematics and its Applications, vol. 42, Cambridge University Press, Cambridge, 1993.

[Krö10] Bernhard Krön, *Cutting up graphs revisited—a short proof of Stallings' structure theorem*, Groups Complex. Cryptol. **2** (2010), no. 2, 213–221.

[Mar16] David Marker, *Lectures on infinitary model theory*, Lecture Notes in Logic, vol. 46, Cambridge University Press, Cambridge, 2016.

[Rol98] Martin Roller, *Poc sets, median algebras and group actions*, Ph.D. thesis, Universität Regensburg, 1998.

[Tse20] Anush Tserunyan, *A descriptive construction of trees and Stallings' theorem*, Trends in set theory, Contemp. Math., vol. 752, Amer. Math. Soc., Providence, RI, 2020, pp. 191–207.