On the large section uniformization theorem

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In this note, we give a proof of the following standard result, which is a trivial rephrasing of the original proof, but with a more "topological" and less "combinatorial" flavor.

Theorem (Kechris [Kec95*, 18.6*])**.** Let $f : X \to Y$ be a Borel map between standard Borel spaces, and $Y \ni y \mapsto \mathcal{I}_y \subseteq \mathcal{B}(f^{-1}(y))$ be an assignment of a σ-ideal of Borel sets on each fiber of f. For $A \in \mathcal{B}(X)$, put

$$\exists^{\mathcal{I}}_f(A) := \{y \in Y \mid f^{-1}(y) \cap A \notin \mathcal{I}_y\}.$$

Assume that $\exists^{\mathcal{I}}_f(A) \in \mathbf{\Delta}^1_1(f(X))$ for each $A \in \mathcal{B}(X)$, and that $\exists^{\mathcal{I}}_f(X) = f(X)$. Then $f(X) \subseteq Y$ is Borel, and there is a Borel section $g : f(X) \hookrightarrow X$ of f, meaning $f \circ g = \mathrm{id}_{f(X)}$.

Proof. Fix a compatible Polish topology on X such that each fiber $f^{-1}(y)$ is closed, and let $\mathcal{F}(X)$ denote the Effros Borel space. For $y \in Y$, let $\mathrm{supp}_{\mathcal{I}}(y) \subseteq f^{-1}(y)$ be the **closed support** of $\mathcal{I}_y$, i.e., the smallest closed subset whose complement is in $\mathcal{I}_y$. Then for open $U \subseteq X$,

$$\mathrm{supp}_{\mathcal{I}}(y) \cap U \neq \varnothing \iff y \in \exists^{\mathcal{I}}_f(U);$$

thus $\mathrm{supp}_{\mathcal{I}} : f(X) \to \mathcal{F}(X)$ is $\mathbf{\Delta}^1_1$-measurable. It follows by Lusin separation that $\mathrm{supp}_{\mathcal{I}}$ extends to a Borel map $s : Y \to \mathcal{F}(X)$ (indeed, we may assume without loss of generality that $\mathcal{F}(X) \cong 2^{\mathbb{N}}$; then for each subbasic clopen $U_i = \{z \mid z(i) = 1\} \subseteq 2^{\mathbb{N}}$, the $\mathbf{\Sigma}^1_1$ sets $\mathrm{supp}_{\mathcal{I}}^{-1}(U_i), f(X) \setminus \mathrm{supp}_{\mathcal{I}}^{-1}(U_i)$ may be separated by a Borel set, whose indicator function is the ith coordinate of the desired s). Now let $g : Y \to X$ be the composition of s followed by the Kuratowski–Ryll-Nardzewski selector $\mathcal{F}(X) \to X$ (see [Kec95, 12.13]), so that $g(y) \in s(y)$ whenever $s(y) \neq \varnothing$. Then

$$f(X) = \{y \in Y \mid f(g(y)) = y\};$$

indeed, $\subseteq$ holds because for $y \in f(X)$, we have $s(y) = \mathrm{supp}_{\mathcal{I}}(y) \in \mathcal{F}(f^{-1}(y)) \setminus \{\varnothing\}$ since $\exists^{\mathcal{I}}_f(X) = f(X)$, while $\supseteq$ is obvious. Thus $f(X)$ is Borel and g restricted to $f(X)$ is a section of f. ☐

Corollary (Lusin–Suslin)**.** Let $f : X \to Y$ be a Borel injection between standard Borel spaces. Then $f(X) \subseteq Y$ is Borel.

Proof. Put the trivial σ-ideal $\mathcal{I}_y := \{\varnothing\}$ on each fiber $f^{-1}(y)$; then for Borel $A \subseteq X$, $\exists^{\mathcal{I}}(f(A)) = f(A) \subseteq f(X)$ is $\mathbf{\Delta}^1_1$ since its complement in $f(X)$ is $f(X \setminus A)$. ☐

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References

[Kec95] A. S. Kechris, *Classical descriptive set theory*, Graduate Texts in Mathematics, vol. 156, Springer-Verlag, New York, 1995.

[Kec95*] A. S. Kechris, *Classical descriptive set theory: corrections and updates*, https://pma.caltech.edu/documents/5656/CDST-corrections.pdf.