On the Lopez-Escobar theorem for marked structures

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The classical Lopez-Escobar theorem [LE65], [Vau75] shows that a Borel, isomorphism-invariant set in the Polish space of structures on a fixed countable set is $\mathcal{L}_{\omega_1\omega}$-definable. In [Che25], we gave an axiomatic account of general "topological spaces parametrizing countable structures" obeying the usual descriptive set-theoretic properties, and proved a version of Lopez-Escobar in that context. Other than the classical space of structures on $\mathbb{N}$, one commonly used such "space of structures" consists of *marked structures* (e.g., *marked groups*) over a fixed generating set. In this note, we give a brief, self-contained exposition of the proof of Lopez-Escobar in this special case.

Let $\mathcal{L}$ be a countable functional first-order language, $V = \{v_0, v_1, v_2, \dots\}$ be a countably infinite set of variables, and $\langle V \rangle$ be the set of first-order $\mathcal{L}$-terms generated by V.

The **space of marked $\mathcal{L}$-structures** is the space of $\mathcal{L}$-congruence relations on $\langle V \rangle$:

$$X_0 := \left\{ \sim \in 2^{\langle V \rangle^2} \;\middle|\; \begin{array}{c} \forall t \in \langle V \rangle\, (t \sim t), \\ \forall s, t \in \langle V \rangle\, (s \sim t \implies t \sim s), \\ \forall r, s, t \in \langle V \rangle\, (r \sim s \sim t \implies r \sim t), \\ \forall n\text{-ary } f \in \mathcal{L}\, \forall s_0, \dots, s_{n-1}, t_0, \dots, t_{n-1} \in \langle V \rangle \left(\bigwedge_i (s_i \sim t_i) \implies f(\vec{s}) \sim f(\vec{t}) \right) \end{array} \right\},$$

a zero-dimensional Polish space. We think of $\sim \in X_0$ as coding the quotient structure $\langle V \rangle / {\sim}$.

For each $n \in \mathbb{N}$, the **space of n-pointed marked $\mathcal{L}$-structures** is

$$\begin{aligned} X_n &:= \left\{ (\sim, [t_0], \dots, [t_{n-1}]) \;\middle|\; \sim \in X_0 \text{ and } [t_0], \dots, [t_{n-1}] \in \langle V \rangle / {\sim} \right\} \\ &\cong (X_0 \times \langle V \rangle^n) / {\approx} \qquad \text{where } (\sim, \vec{s}) \approx (\sim, \vec{t}) \;:\iff\; \forall i\, (s_i \sim t_i). \end{aligned}$$

This is also a zero-dimensional Polish space, since we may exhibit X_n as a retract of $X_0 \times \langle V \rangle^n$ by mapping each congruence class to its least element in some arbitrarily chosen enumeration of $\langle V \rangle$. Note that $\approx$ is relatively open in the fiber product $(X_0 \times \langle V \rangle^n) \times_{X_0} (X_0 \times \langle V \rangle^n) \cong X_0 \times \langle V \rangle^{2n}$; thus the quotient map $X_0 \times \langle V \rangle^n \twoheadrightarrow X_n$ is open. It follows that X_n has an open basis consisting of images of basic open rectangles in $X_0 \times \langle V \rangle^n$, which are of the form

$$U_{\phi, \vec{s}, \vec{t}} := \left\{ (\sim, [\vec{s}]) \;\middle|\; \langle V \rangle / {\sim} \models \phi([\vec{t}]) \right\}$$

where $\vec{s} \in \langle V \rangle^n$, $\vec{t} \in \langle V \rangle^m$, and $\phi(x_0, \dots, x_{m-1})$ is a finite conjunction of equations and inequations.

The **isomorphism groupoid of marked $\mathcal{L}$-structures** is

$$\begin{aligned} G &:= \left\{ (\sim, \sim', g) \;\middle|\; \sim, \sim' \in X_0 \text{ and } g : \langle V \rangle / {\sim} \cong \langle V \rangle / {\sim'} \right\} \\ &\cong \left\{ (\sim, \sim', \tilde{g}) \in X_0 \times X_0 \times 2^{\langle V \rangle^2} \;\middle|\; \tilde{g} \text{ is } (\sim \times \sim')\text{-invariant and descends to an isomorphism} \right\}. \end{aligned}$$

This is also a zero-dimensional Polish space. The topology is generated by the first and second coordinate projections $\mathrm{dom}, \mathrm{cod} : G \to X_0$ as well as the subbasic open sets

$$[\![s \mapsto t]\!] := \{ (\sim, \sim', g) \mid g([s]_\sim) = [t]_{\sim'} \}$$

for fixed $s, t \in \langle V \rangle$; note that it is not necessary to include complements of these sets in the subbasis, since g is a function and so $g([s]) \neq [t] \iff \exists t' \not\sim' t\, (g([s]) = [t'])$.

For each n, we have a natural action

$$\begin{aligned} \cdot : G \times_{X_0} X_n &\longrightarrow X_n \\ ((\sim, \sim', g), (\sim, [\vec{a}])) &\longrightarrow (\sim', g([\vec{a}])). \end{aligned}$$

Lemma. For every open set $U \subseteq X_n$, the isomorphism saturation $G \cdot U \subseteq X_n$ is Σ_1-definable. (Here by a Σ_1 formula, we mean a countable disjunction of finitary existential formulas.)

Proof. For a basic open $U = U_{\phi,\vec{s},\vec{t}}$ as above, where $\vec{s} \in \langle V \rangle^n$, $\vec{t} \in \langle V \rangle^m$, and $\phi(x_0, \dots, x_{m-1})$ is finite quantifier-free, let $k \in \mathbb{N}$ be large enough so that the terms $\vec{s}, \vec{t}$ only mention the variables $v_0, \dots, v_{k-1} \in V$; we claim that $G \cdot U_{\phi,\vec{s},\vec{t}}$ is defined by the formula

$$\psi(x_0, \dots, x_{n-1}) := \exists v_0, \dots, v_{k-1} \big((x_0 = s_0) \wedge \dots \wedge (x_{n-1} = s_{n-1}) \wedge \phi(\vec{t})\big).$$

Indeed, for every $(\sim, [\vec{s}]) \in U_{\phi,\vec{s},\vec{t}}$, the tuple $[\vec{s}] \in (\langle V \rangle/{\sim})^n$ clearly satisfies ψ, with each v_i witnessed by its own congruence class $[v_i] \in \langle V \rangle/{\sim}$. Conversely, if $(\sim', [\vec{s}']) \in X_n$ is such that $\langle V \rangle/{\sim'} \models \psi([\vec{s}'])$, then there is an assignment of witnesses $\tilde{g} : \langle v_0, \dots, v_{k-1} \rangle \to \langle V \rangle/{\sim'}$ such that $[s'_i] = \tilde{g}(s_i)$ and $\langle V \rangle/{\sim'} \models \phi(\tilde{g}(\vec{t}))$. Extend $\tilde{g}$ to the rest of the variables arbitrarily to get a surjective homomorphism $\tilde{g} : \langle V \rangle \twoheadrightarrow \langle V \rangle/{\sim'}$, and let $\sim$ be the congruence kernel of $\tilde{g}$; then $\tilde{g}$ descends to an isomorphism $g : \langle V \rangle/{\sim} \cong \langle V \rangle/{\sim'}$ such that $[\vec{s}] = g([\vec{s}])$ and $\langle V \rangle/{\sim'} \models \phi(g([\vec{t}]))$, whence $\langle V \rangle/{\sim} \models \phi([\vec{t}])$, whence $(\sim, [\vec{s}]) \in U_{\phi,\vec{s},\vec{t}}$ is isomorphic to $(\sim', [\vec{s}'])$. □

Now for an arbitrary set $A \subseteq X_n$, define the **Vaught transform**

$$G * A := \big\{(\sim, [\vec{t}]) \in X_n \;\big|\; \exists^* g \in \mathrm{cod}^{-1}(\sim) \big((\sim, [\vec{t}]) \in g \cdot A\big)\big\}.$$

Note that for A open, this is $G \cdot A$, since $\mathrm{cod}^{-1}(\sim) \subseteq G$ is a Polish space. More generally, this holds as long as $\{g \mid (\sim, [\vec{t}]) \in g \cdot A\} = \{h \mid h \cdot (\sim, [\vec{t}]) \in A\}^{-1}$ is open for each $(\sim, [\vec{t}]) \in X_n$; we call such A **orbitwise open**. This includes in particular isomorphism-invariant A, for which $G * A = G \cdot A = A$.

Theorem. For every $\mathbf{\Sigma}^0_\alpha$ set $A \subseteq X_n$, $G * A \subseteq X_n$ is Σ_α-definable.

In particular, if A is isomorphism-invariant, then A is Σ_α-definable.

Proof. By induction on the Borel complexity of A. For $\alpha = 1$, this is the preceding lemma. For a countable union of A's, we may take the disjunction of the defining formulas. Thus, it remains to show, assuming the result is true for a given α and all n, A, that $G * B$ for a $\mathbf{\Pi}^0_\alpha$ set $B = X_n \setminus A$ is $\Sigma_{\alpha+1}$-definable. We have

$$(\sim, [\vec{t}]) \in G * B \iff \exists^* g \in \mathrm{cod}^{-1}(\sim) \big((\sim, [\vec{t}]) \in g \cdot B\big).$$

A countable open basis for $\mathrm{cod}^{-1}(\sim)$ is given by its intersections with finitely many $[\![r_i \mapsto s_i]\!]$, for $r_i, s_i \in \langle V \rangle$; thus by the Baire property, the above is

$$\begin{aligned} &\iff \exists m\, \exists \vec{r}, \vec{s} \in \langle V \rangle^m \begin{pmatrix} \mathrm{cod}^{-1}(\sim) \cap \bigcap_i [\![r_i \mapsto s_i]\!] \neq \varnothing \text{ and} \\ \forall^* g \in \mathrm{cod}^{-1}(\sim) \cap \bigcap_i [\![r_i \mapsto s_i]\!] \big((\sim, [\vec{t}]) \in g \cdot B\big) \end{pmatrix} \\ &\iff \exists m\, \exists \vec{r}, \vec{s} \in \langle V \rangle^m \begin{pmatrix} \exists g \in \mathrm{cod}^{-1}(\sim) \cap \bigcap_i [\![r_i \mapsto s_i]\!] \big((\sim, [\vec{t}]) \in g \cdot X_n\big) \\ \wedge \neg \exists^* g \in \mathrm{cod}^{-1}(\sim) \cap \bigcap_i [\![r_i \mapsto s_i]\!] \big((\sim, [\vec{t}]) \in g \cdot A\big) \end{pmatrix}. \end{aligned}$$

Now note that $g \in \mathrm{cod}^{-1}(\sim)$ takes $[r_i] \mapsto [s_i]$, and takes some element of A to $[\vec{t}]$, iff it takes some element of $A \times \{[\vec{r}]\}$ to $[(\vec{t}, \vec{s})]$; thus the above becomes

$$\begin{aligned}
&\iff \exists m\, \exists \vec{r}, \vec{s} \in \langle V \rangle^m \begin{pmatrix} \exists g \in \mathrm{cod}^{-1}(\sim) \left((\sim, [(\vec{t}, \vec{s})]) \in g \cdot (X_n \times \{[\vec{r}]\})\right) \\ \wedge \neg \exists^* g \in \mathrm{cod}^{-1}(\sim) \left((\sim, [(\vec{t}, \vec{s})]) \in g \cdot (A \times \{[\vec{r}]\})\right) \end{pmatrix} \\
&\iff \langle V \rangle / {\sim} \models \bigvee_m \bigvee_{\vec{r} \in \langle V \rangle^m} \exists \vec{y} \left(\phi_{\vec{r}}([\vec{t}], \vec{y}) \wedge \neg \psi_{\vec{r}}([\vec{t}], \vec{y})\right)
\end{aligned}$$

where $\phi_{\vec{r}}, \psi_{\vec{r}}$ are Σ_α formulas from the induction hypothesis. □

Remark. The above spaces X_0, X_1, X_n are X, M, M_X^n from [Che25, Example 6.12], applied to a Morleyization of the original language to make negated atomic formulas Σ_1 [Che25, Example 3.4].

References

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