On the Lusin–Sierpiński theorem and ccc σ-ideals

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It is a classical result that every analytic set in a standard Borel space is universally measurable and ω-universally Baire. In fact, the argument is not specific to measure or category, and applies to show approximability of analytic by Borel sets modulo any σ-ideal of Borel sets with the countable chain condition (ccc). In this note, we give an exposition of this result, following [KS].

Let X be a standard Borel space, $T \subseteq X \times \mathbb{N}^{<\omega}$ be Borel such that each $T_x := \{t \mid (x,t) \in T\}$ is a subtree of $\mathbb{N}^{<\omega}$, $[T] := \{(x,y) \in X \times \mathbb{N}^{\mathbb{N}} \mid \forall n\, (y|_n \in T_x)\}$ be the space of infinite branches in each fiber, and $p : T \sqcup [T] \to X$ be the first coordinate projection. Thus every analytic set $A \subseteq X$ is $p([T])$ for some such T; and by a standard topologization argument, every Borel map $f : Y \to X$ from another standard Borel space Y is isomorphic to $p : [T] \to X$ for some such T.

Let

$$T' := \{(x,t) \in T \mid \exists i \in \mathbb{N}\, ((x,(t,i)) \in T)\}$$

be the one-step pruning, and define as usual the iterated prunings

$$\begin{aligned} T^{(0)} &:= T, \\ T^{(\alpha+1)} &:= T^{(\alpha)\prime}, \\ T^{(\alpha)} &:= \textstyle\bigcap^{\downarrow}_{\beta<\alpha} T^{(\beta)} \quad \text{for a limit ordinal } \alpha. \end{aligned}$$

Then $T^{(\alpha)} \subseteq X \times \mathbb{N}^{<\omega}$ is Borel for all $\alpha < \omega_1$, and we have

$$T \supseteq T' \supseteq T'' \supseteq \cdots \supseteq T^{(\omega)} \supseteq T^{(\omega+1)} \supseteq \cdots$$

with $[T^{(\alpha)}] = [T]$ for each α. Since each T_x is countable, this sequence stabilizes by some $\alpha \le \omega_1$. Each inclusion $T_x^{(\alpha)} \supseteq T_x^{(\alpha+1)}$ is proper iff $T_x^{(\alpha)}$ is (nonempty and) not yet pruned; thus

$$p(T \setminus T') \supseteq p(T' \setminus T'') \supseteq \cdots \supseteq p(T^{(\omega)} \setminus T^{(\omega+1)}) \supseteq \cdots$$

with $\bigcap^{\downarrow}_{\alpha<\omega_1} p(T^{(\alpha)} \setminus T^{(\alpha+1)}) = \varnothing$. Since pruned trees always have branches, we have

$$p(T^{(\alpha)}) \setminus p(T^{(\alpha)} \setminus T^{(\alpha+1)}) = p([T]) \setminus p(T^{(\alpha)} \setminus T^{(\alpha+1)}).$$

Theorem (Lusin–Sierpiński)**.** Let X be a standard Borel space. Every analytic set $A \subseteq X$ is a ω_1-intersection $\bigcap^{\downarrow}_{\alpha<\omega_1} B_\alpha$ and a ω_1-union $\bigcup^{\uparrow}_{\alpha<\omega_1} C_\alpha$ of Borel sets $B_\alpha, C_\alpha \subseteq X$.

Proof. Represent A as $p([T])$ for a Borel family of trees $T \subseteq X \times \mathbb{N}^{<\omega}$ as above; then

$$\begin{aligned} A = (T^{(\omega_1)})^{\varnothing} &= \textstyle\bigcap^{\downarrow}_{\alpha<\omega_1} (T^{(\alpha)})^{\varnothing} = \bigcap^{\downarrow}_{\alpha<\omega_1} p(T^{(\alpha)}) \\ &= p([T]) \setminus \textstyle\bigcap^{\downarrow}_{\alpha<\omega_1} p(T^{(\alpha)} \setminus T^{(\alpha+1)}) \\ &= \textstyle\bigcup^{\uparrow}_{\alpha<\omega_1} (p([T]) \setminus p(T^{(\alpha)} \setminus T^{(\alpha+1)})) \\ &= \textstyle\bigcup^{\uparrow}_{\alpha<\omega_1} (p(T^{(\alpha)}) \setminus p(T^{(\alpha)} \setminus T^{(\alpha+1)})). \end{aligned}$$

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Remark. If $T^{(\alpha)}$ above is already pruned for some $\alpha < \omega_1$, then $A = p(T^{(\alpha)}) \subseteq X$ is Borel, and moreover $X \ni x \mapsto [T_x] = [T_x^{(\alpha)}] \in \mathcal{F}(\mathbb{N}^{\mathbb{N}})$ is Borel, whence $p : [T] \twoheadrightarrow A$ has a Borel section by the Kuratowski–Ryll-Nardzewski selection theorem (i.e., choose the leftmost branch through each $T_x^{(\alpha)}$).

Working modulo a ccc σ-ideal, everything must stabilize by some $\alpha < \omega_1$, yielding

Theorem (classical)**.** Let X be a standard Borel space, $\mathcal{I} \subseteq \mathcal{B}(X)$ be a ccc σ-ideal of Borel sets. For any analytic $A = f(Y) \subseteq X$, where $f : Y \to X$ is a Borel map from some other standard Borel space, there is a $N \in \mathcal{I}$ such that $A \setminus N$ is Borel and there is a Borel section $s : A \setminus N \to Y$ of f.

Proof. We may assume $Y = [T]$ for some Borel family of trees $T \subseteq X \times \mathbb{N}^{<\omega}$ as above and f is the projection $p : [T] \to X$. Let $p^*(\mathcal{I}) \subseteq \mathcal{B}(T)$ be the σ-ideal of all Borel $B \subseteq T$ such that $p(B) \in \mathcal{I}$. Using that $p : T \to X$ is countable-to-1, $p^*(\mathcal{I})$ is easily seen to be ccc. Thus the sequence of $T^{(\alpha)}$'s must stabilize mod $p^*(\mathcal{I})$ by some $\alpha < \omega_1$, i.e., $T^{(\alpha)} \setminus T^{(\alpha+1)} \in p^*(\mathcal{I})$. Put $N := p(T^{(\alpha)} \setminus T^{(\alpha+1)})$. Then as noted above, $A \setminus N = p([T]) \setminus p(T^{(\alpha)} \setminus T^{(\alpha+1)}) = p(T^{(\alpha)}) \setminus p(T^{(\alpha)} \setminus T^{(\alpha+1)})$ is Borel; and since every fiber of $T^{(\alpha)}$ over this set is pruned, we may find the Borel section s as in the preceding remark by selecting the leftmost branch in each $T_x^{(\alpha)}$. □

Remark. In the above proof, we have $A \cup N = p(T^{(\alpha)}) \cup p(T^{(\alpha)} \setminus T^{(\alpha+1)}) = p(T^{(\alpha)})$. Thus the proof essentially shows that a pair of sets appearing in the ω_1-sequences in the Lusin–Sierpiński theorem (which sandwich A) have difference belonging to $\mathcal{I}$.

We may also interpret the above as a disjointness result for σ-ideals. For an arbitrary set $A \subseteq X$, define the **Borel ideal** (or "Borelness ideal") $\mathrm{BOREL}(A)$ to be the set of all Borel $B \subseteq X$ such that $A \cap B$ is Borel. Then the above says that for analytic A, $\mathrm{BOREL}(A)$ is mutually singular with every ccc σ-ideal, as witnessed by the fact that $\mathrm{BOREL}(A)$ contains the σ-ideal generated by $\{X \setminus p(T^{(\alpha)} \setminus T^{(\alpha+1)})\}_{\alpha<\omega_1}$ which is in turn mutually singular with every ccc σ-ideal.

Acknowledgments I would like to thank Forte Shinko and Felix Weilacher for inspiring discussions about this topic and for making me aware of the paper [KS].

References

[KS] A. S. Kechris and S. Solecki, *Approximation of analytic by Borel sets and definable countable chain conditions*, Israel J. Math. **89** (1995), 343–356.