On the Lusin–Suslin and Lusin unicity theorems

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In this note, we give simple proofs of the following classical results of Lusin–Suslin and Lusin; see [Kec, 15.1, 18.11]. In the proof, for maximum clarity, we take the definition of **standard Borel space** to be a measurable space which is isomorphic to a Borel subset of $2^{\mathbb{N}}$. See the subsequent remarks for comments on this, as well as connections between this proof and other results.

Theorem (generalized Lusin–Suslin)**.** Let $f : X \to Y$ be a Borel map between standard Borel spaces, $Z \subseteq Y$ be an arbitrary subset such that $f|_{f^{-1}(Z)}$ is injective. Then $f(X) \cap Z \in \mathbf{\Delta}^1_1(Z)$.

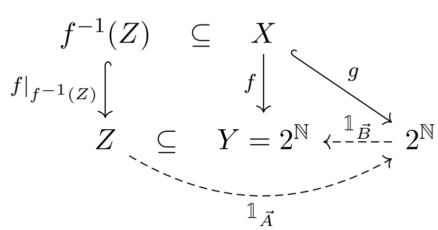
Proof. We may assume $Y = 2^{\mathbb{N}}$. Since X is standard Borel, there is a Borel embedding $g : X \hookrightarrow 2^{\mathbb{N}}$ with Borel image. For each subbasic clopen $U_i := \{\vec{x} \mid x_i = 1\} \subseteq 2^{\mathbb{N}}$, there are Borel $B_i \subseteq 2^{\mathbb{N}}$ as well as $A_i \in \mathbf{\Delta}^1_1(Z)$ such that

$$g^{-1}(B_i) = f^{-1}(U_i), \qquad\qquad f^{-1}(A_i) = g^{-1}(U_i) \cap f^{-1}(Z),$$

the former since f is Borel and g is a Borel embedding, and the latter by applying $\mathbf{\Pi}^1_1$ reduction to the $\mathbf{\Pi}^1_1$ sets $Y \setminus f(g^{-1}(U_i)), Y \setminus f(X \setminus g^{-1}(U_i)) \subseteq Y$ which cover Z since $f|_{f^{-1}(Z)}$ is injective. This means the indicator functions $\mathbb{1}_{\vec{B}} = (\mathbb{1}_{B_i})_i : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ and $\mathbb{1}_{\vec{A}} = (\mathbb{1}_{A_i})_i : Z \to 2^{\mathbb{N}}$ obey

$$(*) \qquad \mathbb{1}_{\vec{B}} \circ g = f, \qquad\qquad \mathbb{1}_{\vec{A}} \circ f|_{f^{-1}(Z)} = g|_{f^{-1}(Z)}$$

(see the following diagram):



Then

$$f(X) \cap Z = \{z \in Z \mid z = \mathbb{1}_{\vec{B}}(\mathbb{1}_{\vec{A}}(z)) \ \& \ \mathbb{1}_{\vec{A}}(z) \in g(X)\}.$$

Indeed, $\subseteq$ is obvious from $(*)$; conversely, if z belongs to this set, then $\mathbb{1}_{\vec{A}}(z) = g(x)$ for some $x \in X$, whence $z = \mathbb{1}_{\vec{B}}(\mathbb{1}_{\vec{A}}(z)) = \mathbb{1}_{\vec{B}}(g(x)) = f(x)$. And this set is $\mathbf{\Delta}^1_1(Z)$, since $\mathbb{1}_{\vec{A}}$ is $\mathbf{\Delta}^1_1$-measurable. $\square$

Corollary (Lusin–Suslin)**.** Let $f : X \to Y$ be an injective Borel map between standard Borel spaces. Then $f(X) \subseteq Y$ is Borel.

Proof. Take $Z = Y$ above. $\square$

Corollary (Lusin)**.** Let $f : X \to Y$ be an arbitrary Borel map between standard Borel spaces. Then $\exists!_f(X) := \{y \in Y \mid |f^{-1}(y)| = 1\} \subseteq Y$ is $\mathbf{\Pi}^1_1$.

Proof. Take $Z = \{y \in Y \mid |f^{-1}(y)| \leq 1\}$ above. $\square$

Remark. The above statements imply that for Borel $R \subseteq X \times Y$, $\exists!_Y(R) := \{x \in X \mid \exists! y\, R(x,y)\}$ is $\mathbf{\Pi}^1_1$, respectively Borel if the existence is always unique, by taking the projection $f : R \to X$.

Remark. As noted above, we took "standard Borel space" to mean a measurable space which is Borel isomorphic to a Borel subset of $2^{\mathbb{N}}$ (with the induced σ-algebra). This is easily seen to be equivalent to the usual definition of being induced by a Polish topology, via a change of topology argument. More precisely, note that such an argument preserves the Borel σ-algebra by definition, without using Lusin–Suslin (which would render the above proof of Lusin–Suslin circular).

Remark. The above proof of Lusin–Suslin, with $\mathbf{\Sigma}^1_1$ separation in place of $\mathbf{\Pi}^1_1$ reduction when f is injective, is in some sense essentially equivalent to that given in [Ch2].

In that proof, we replaced the latter half of the above argument with the topological analogue of Lusin–Suslin, namely that a topological embedding between Polish spaces has $\mathbf{\Pi}^0_2$ image. This topological result may in turn be proved via an argument like the latter half of the above; see [Ch1, 4.1]. As explained there, both this result and the latter half of Lusin–Suslin (that a Borel *embedding* between standard Borel spaces has Borel image) amount to the dual algebraic fact that any presentation $\langle G \mid R \rangle$ of an algebraic structure, e.g., a finite presentation of a group, may be turned into a presentation with any other generating set H obeying the same cardinality bound.

The above proof of Lusin's theorem is also inspired in part by the classical proof in [Kur, §39 VII] using "Lusin schemes", which may likewise be seen as generalizing the "Lusin scheme" proof of Lusin–Suslin [Kec, 15.1].

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References

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